

# Co-periodic cyclic homology

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## Introduction.

Periodic cyclic homology  $HP_{\bullet}(A)$  of a unital associative flat algebra  $A$  over a commutative ring  $k$  is defined as the homology of an explicit complex — or rather, a bicomplex  $CP_{\bullet,\bullet}(A)$ . In one direction, the bicomplex is bounded: one has  $CP_{i,\bullet}(A) = 0$  for  $i < 0$ . However, there are no restrictions in the other direction. On the contrary,  $CP_{i,j}(A)$  is 2-periodic with respect to the index  $j$ . Therefore turning the bicomplex  $CP_{\bullet,\bullet}(A)$  into a complex requires a choice.

In the standard theory, what one considers is the product-total complex of  $CP_{\bullet,\bullet}(A)$ . Traditionally, considering the sum-total complex was assumed to be a wrong thing to do, since if  $k$  contains  $\mathbb{Q}$ , the sum-total complex is trivially acyclic (see e.g. [L, Section 5.1.2]). Nevertheless, about ten years ago, it has been suggested by M. Kontsevich ([Ko1], [Ko2, 2.32]) that if  $k$  is a field of positive characteristic  $p$ , then taking the sum-total complex gives a meaningful and interesting new theory.

It seems that Kontsevich's suggestion was not taken seriously at the time (in particular, I didn't follow it up in [Ka1] and [Ka2] — in retrospect, a rather glaring omission). However, recently a very similar phenomenon has appeared in the work of A. Beilinson and B. Bhatt. They work in the commutative situation, where periodic cyclic homology is known to be intimately related to the de Rham cohomology. In particular, if  $A$  is a commutative algebra over a field  $k$  of characteristic 0 with a smooth spectrum  $X = \operatorname{Spec} A$ , then  $HP_{\bullet}(A)$  is entirely expressible in terms of de Rham cohomology groups  $H_{DR}^{\bullet}(A) = H_{DR}^{\bullet}(X)$ . In the situations considered by Beilinson and Bhatt,  $k$  is a field  $\mathbb{F}_q$  of positive characteristic  $p$  (or more generally, a truncated Witt vectors ring  $W_n(\mathbb{F}_q)$ ,  $n \geq 1$ ), and  $\operatorname{Spec} A$  is not smooth. So, instead of de Rham cohomology, one works with *derived de Rham cohomology* introduced by L. Illusie [I]. To define it for an affine variety  $X = \operatorname{Spec} A$ , one replaces the algebra  $A$  with a simplicial resolution  $A_{\bullet}$  whose terms are free commutative algebras, and takes de Rham complex termwise. Then after passing to the usual standard complex of a simplicial object, one obtains a bicomplex

$\Omega^\bullet(A_\bullet)$ , and here again, there is the issue of how to totalize it. In Illusie's approach, one takes the product-total complex. If we denote its cohomology simply by  $H_{DR}^\bullet(A)$ , then the standard Hodge-to-de Rham spectral sequence induces a convergent spectral sequence

$$\Lambda^\bullet \Omega_\bullet(A) \Rightarrow H_{DR}^\bullet(A),$$

where  $\Lambda^\bullet$  is an appropriate derived version of the exterior power functors, and  $\Omega_\bullet(A)$  is the cotangent complex of  $A$ . In particular,  $H_{DR}^\bullet(A)$  does not depend on the choice of a resolution  $A_\bullet$ .

If one instead takes the sum-total complex of the bicomplex  $\Omega^\bullet(A_\bullet)$  and denotes its homology by  $\overline{H}_{DR}^\bullet(A)$ , then a different spectral sequence becomes convergent — namely, the spectral sequence

$$H_{DR}^\bullet(A_\bullet) \Rightarrow \overline{H}_{DR}^\bullet(A).$$

If  $k$  is a field of characteristic 0, this shows that  $\overline{H}_{DR}^\bullet(A)$  is completely uninteresting. Indeed, since the terms  $A_i$  of the resolution  $A_\bullet$  are free algebras, so that  $\text{Spec } A_i$  are just affine spaces, the de Rham cohomology  $H_{DR}^\bullet(A_i)$  is simply  $k$  placed in degree 0, irrespective of  $A_i$ , and one deduces that  $\overline{H}_{DR}^\bullet(A)$  is also  $k$  placed in degree 0.

However, the situation changes drastically if  $k$  is a field of positive characteristic. In this case, affine spaces have lots of de Rham cohomology — in fact, one has the classic Cartier isomorphism

$$H_{DR}^i(A_j) \cong \Omega^i(A_j^{(1)}),$$

where (1) indicates the Frobenius twist. With this in mind, what we obtain is the derived version of the *conjugate spectral sequence*, and it reads as

$$\Lambda^\bullet \Omega_\bullet(A^{(1)}) \Rightarrow \overline{H}_{DR}^\bullet(A).$$

In particular,  $\overline{H}_{DR}^\bullet(A)$  also does not depend on the resolution  $A_\bullet$ , and this justifies our notation. Beilinson and Bhatt go on to apply this observation to some specific algebras  $A$  important in  $p$ -adic Hodge theory, where working with  $\overline{H}_{DR}^\bullet(-)$  instead of  $H_{DR}^\bullet(-)$  allows one to obtain much stronger results.

The goal of the present paper is then the following. Motivated by the work of Beilinson and Bhatt, we take up Kontsevich's original suggestion and study “wrong” totalizations of the cyclic bicomplex  $CP_{\bullet,\bullet}(A)$ . To make room for interesting applications, we work with DG algebras  $A_\bullet$  instead of just associative algebras. For any DG algebra  $A_\bullet$  over a commutative

ring  $k$ , we define a functorial *co-periodic cyclic complex*  $\overline{CP}_\bullet(A_\bullet)$ , and we call its homology  $\overline{HP}_\bullet(A_\bullet)$  *co-periodic cyclic homology* of  $A_\bullet$ . If  $A_\bullet = A$  is concentrated in homological degree 0, then  $\overline{CP}_\bullet(A)$  is literally the sum-total complex of the periodic cyclic bicomplex  $CP_{\bullet,\bullet}(A)$ , and in the general case, the definition requires only a minor modification. We then extend the definition to small DG categories  $A_\bullet$ , and we prove the following:

- As soon as the base ring  $k$  is Noetherian,  $\overline{HP}_\bullet(A_\bullet)$  is derived Morita-invariant, and moreover, gives an additive invariant of small DG categories in the sense of [Ke].
- If  $k$  is a field of finite characteristic, and  $A_\bullet$  is a cohomologically bounded smooth DG algebra over  $k$ , then  $\overline{HP}_\bullet(A_\bullet) \cong HP_\bullet(A_\bullet)$ .

Our actual statements, Theorem 6.5, Theorem 6.6 and Theorem 6.7, are slightly stronger and more precise, but the above sums up the essential points. We also prove the following somewhat more technical result.

- Assume that  $k$  is a field of odd positive characteristic  $p > 2$ . Then for any small DG category  $A_\bullet$ , there exists a functorial *conjugate spectral sequence*

$$HH_\bullet(A_\bullet)^{(1)}((u^{-1})) \Rightarrow \overline{HP}_\bullet(A_\bullet),$$

where  $HH_\bullet(A_\bullet)$  is the Hochschild homology of  $A_\bullet$ , (1) stands for the Frobenius twist,  $u$  is a formal generator of cohomological degree 2, and  $((u^{-1}))$  is the shorthand for “formal Laurent powers series in  $u^{-1}$ ”.

The conjugate spectral sequence generalizes the usual commutative spectral sequence in the same way that the Hodge-to-de Rham spectral sequence is generalized to the “Hochschild-to-cyclic” spectral sequence

$$HH_\bullet(A_\bullet)((u)) \Rightarrow HP_\bullet(A_\bullet).$$

Note that for the conjugate spectral sequence, we need to take power series in  $u^{-1}$  rather than  $u$ .

The paper is organized as follows. Unfortunately, our definition of co-periodic cyclic homology works by an explicit complex, and we have not been able to find a more invariant treatment similar to [C]. We do rely on Connes’ notion of a cyclic object and the small category  $\Lambda$  to encode all the relevant combinatorics, but we are unable to use the machinery of derived categories. As a next best choice, we equip everything in sight with a filtration, and use filtered derived categories. This is still better than writing down explicit maps of complexes, but it requires a lot of preliminaries.

These are contained in Section 1, Section 2 and Section 3. Almost nothing in these three sections is new, with a possible exception of some (but not all) material in Subsection 1.3, Subsection 2.3 and Subsection 3.4. However, these are things that need to be spelled out in all the gory detail, in order to fix notation and terminology and avoid ambiguities. Section 1 is devoted to filtrations, filtered derived categories, convergence of spectral sequences and suchlike. The possibly new ingredient is a generalization of the canonical truncation to filtered complexes given in Subsection 1.3. Section 2 is devoted to mixed complexes: we state the necessary definitions, discuss examples arising from Tate cohomology of cyclic groups, and prove one simple technical statement on what we call “mixed resolutions” that is used later in the paper. Section 3 contains the definition of the co-periodic cyclic complex, and some immediate observations on its properties. The input here is a cyclic object or a complex of cyclic objects, so that the theory is essentially linear.

The technical heart of the paper are Section 4 and Section 5. We still work in the setting of complexes of cyclic objects. In Section 4, we start using our main technical gadget, namely, a certain version  $\Lambda_p$  of the cyclic category  $\Lambda$  that appeared in [FT]. This is a small category that comes equipped with two natural functors to  $\Lambda$  — an “edgewise subdivision” functor  $i : \Lambda_p \rightarrow \Lambda$ , and a projection  $\pi : \Lambda_p \rightarrow \Lambda$  that corresponds to the  $p$ -fold cover of a circle by itself. After reminding classic results about edgewise subdivision and the functor  $i$ , we give, in Proposition 4.4, a filtered refinement of the main classic result (this refinement seems to be new). Then in Subsection 4.3, we turn to the functor  $\pi$  and prove Lemma 4.7, a kind of a retraction statement that allows to reduce questions about cyclic objects coming from the projection  $\pi$  to questions about their restriction to the subcategory  $\Delta^\circ \subset \Lambda$ .

Section 5 is mostly devoted to the conjugate spectral sequence: we start with a complex of cyclic objects, and we produce a spectral sequence converging to its co-periodic cyclic homology. We need to impose one condition on the complex that we call “tightness”; a version of it already appeared in [Ka3] and before that, in [Ka1]. Then in Subsection 5.4, we turn to the study of convergence of this spectral sequence, and show that convergence questions for cyclic objects can be reduced to questions about their restrictions to  $\Delta^\circ$  — this uses Lemma 4.7 in an essential way.

Finally, in Section 6, we turn to our main subject — namely, DG algebras and DG categories. After giving a brief reminder on DG algebras and DG categories, we define co-periodic cyclic homology of a DG algebra and a small DG category, and prove our main results. The main technical tool here is a very simple fact of linear algebra — namely, the computation

of Tate cohomology of a cyclic group  $\mathbb{Z}/p\mathbb{Z}$  with coefficients in the  $p$ -th tensor power  $V^{\otimes kp}$  of a vector space  $V$  over a field  $k$  of characteristic  $p$ . This has already appeared in [Ka1] and more recently, in [Ka3]. In the present paper, in Subsection 6.3, we give a generalization of this computation to complexes of vector spaces that seems to be new. Together with the material of Section 5, this immediately gives the conjugate spectral sequence and yields Theorem 6.5 and Theorem 6.6. Then the same computation is applied to a certain twisted version of Hochschild homology, and together with the material of Subsection 5.4, allows us to prove Theorem 6.7, our main comparison theorem. In the end, in Subsection 6.5, we explain briefly how our results compare to results about de Rham cohomology, and in particular, to results of Beilinson and Bhatt.

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## 1 Filtrations.

**1.1 Filtered objects.** Assume given an abelian category  $\mathcal{C}$  satisfying  $AB4$  and  $AB4^*$  (that is,  $\mathcal{C}$  has arbitrary products and sums, and both are exact). In this paper, by a *filtration*  $F^\bullet$  on an object  $E \in \mathcal{C}$  we will understand a collection of subobjects  $F^j E \subset E$ , one for each integer  $j$ , such that  $F^j E \subset F^i E$  whenever  $j \geq i$ . In other words, all our filtrations are decreasing and indexed by all integers. A *filtered object* in  $\mathcal{C}$  is an object equipped with a filtration. A filtered object  $\langle E, F^\bullet \rangle$  is *concentrated in a filtered degree*  $i$  if  $F^i E = E$  and  $F^{i+1} E = 0$ . For any filtered object  $\langle E, F^\bullet \rangle$  and any integers  $i \leq j$ , we denote

$$F^{[i,j]} E = F^i E / F^j E,$$

and we denote  $\mathrm{gr}_F^i E = F^{[i,i]} E$ , so that  $\mathrm{gr}_F^\bullet E$  is the associated graded quotient of the filtration  $F^\bullet$ . If  $F$  in  $\mathrm{gr}_F^\bullet$  is clear from the context, we will drop it from notation. For any  $i \leq i' \leq j' \leq j$ , we have a natural commutative

square

$$(1.1) \quad \begin{array}{ccc} F^{[i,j]}E & \longleftarrow & F^{[i',j]}E \\ \downarrow & & \downarrow \\ F^{[i,j]}E & \longleftarrow & F^{[i',j']}E, \end{array}$$

and this square is cartesian and cocartesian. We also have a natural isomorphism

$$(1.2) \quad \lim_{\substack{\rightarrow \\ i}} \lim_{\substack{\leftarrow \\ j}} F^{[i,j]}E \cong \lim_{\substack{\leftarrow \\ j}} \lim_{\substack{\rightarrow \\ i}} F^{[i,j]}E$$

and a natural commutative diagram

$$(1.3) \quad \begin{array}{ccc} E & \xrightarrow{p} & \lim_{\substack{\leftarrow \\ j}} E/F^j E \\ \uparrow i & & \uparrow \\ \lim_{\substack{\rightarrow \\ i}} F^i E & \longrightarrow & \lim_{\substack{\leftarrow \\ j}} \lim_{\substack{\rightarrow \\ i}} F^{[i,j]}E. \end{array}$$

**Definition 1.1.** A filtration  $F^\bullet$  on an object  $E \in \mathcal{C}$  is *convergent*, or equivalently, the object  $E$  is *convergent with respect to  $F^\bullet$* , if all the maps in the diagram (1.3) are isomorphisms. The *completion*  $\widehat{E}$  of a filtered object  $\langle E, F^\bullet \rangle$  in  $\mathcal{C}$  is the object (1.2).

**Example 1.2.** Assume given a graded object  $E = \bigoplus_{i \in \mathbb{Z}} E_i \in \mathcal{C}$ , and filter it by

$$F^i E = \bigoplus_{j \geq i} E_j \subset E.$$

Then in general, this filtration is *not* convergent. The completion  $\widehat{E}$  can be described as

$$\widehat{E} = \prod_{j \geq i} E_j \oplus \bigoplus_{j < i} E_j,$$

where  $i$  is an arbitrary fixed integer (the right-hand side does not depend on the choice of  $i$ ). If all the components  $E_i$  are identified with the same object  $E' \in \mathcal{C}$ , then it is convenient to use shorthand notation

$$E = E'[u, u^{-1}],$$

that is, Laurent polynomials in one formal variable  $u$  of grading degree 1 with coefficients in  $E'$ . Then we have  $\widehat{E} = E'((u))$ , the formal Laurent power series in the same variable  $u$  with the same coefficients.

In general, our terminology is misleading, since there are no natural maps between a filtered object  $E$  and its completion  $\widehat{E}$ . However, say that a filtration  $F^\bullet$  on  $E$  is *exhaustive* if the map  $i$  in (1.3) is an isomorphism. Then  $p$  gives a natural map

$$E \rightarrow \widehat{E} = \lim_{\substack{\leftarrow \\ j}} E/F^j E.$$

Alternatively, say that a filtration  $F^\bullet$  is *bounded below* if  $F^i E = 0$  for  $i \gg 0$ . Then  $p$  is an isomorphism, and  $i$  gives a natural map

$$\widehat{E} = \lim_{\substack{\rightarrow \\ i}} E \rightarrow E.$$

For any filtered object  $\langle E, F^\bullet \rangle$ , the filtration  $F^\bullet$  induces a filtration on the completion  $\widehat{E}$ , and this filtration is convergent. Two different filtrations on the same object can have the same completion; here is one standard situation when this happens.

**Definition 1.3.** Two filtrations  $F_1^\bullet, F_2^\bullet$  on an object  $E \in \mathcal{C}$  are *commensurable* if for any integer  $i$ , there exist integers  $i_1, i_2$  such that  $i_1 \leq i \leq i_2$  and

$$F_1^{i_2} E \subset F_2^i E \subset F_1^{i_1} E, \quad F_2^{i_2} E \subset F_1^i E \subset F_2^{i_1} E.$$

Then for any two commensurable filtrations  $F_1, F_2$  and for any two integers  $i \leq j$ ,  $F_1^{[i,j]} E$  is concentrated in a finite range of filtered degrees with respect to  $F_1$ , so that it is complete, and analogously,  $F_2^{[i,j]} E$  is concentrated in a finite range of filtered degrees with respect to  $F_2$ . The completions of  $E$  with respect to  $F_1$  and  $F_2$  are canonically identified.

For example, for any integers  $n \geq 1, m$ , and filtration  $F^\bullet$ , define the  *$n$ -th rescaling*  $F_{[n]}^\bullet$  and the *shift by  $m$*   $F_m^\bullet$  of the filtration  $F^\bullet$  by

$$(1.4) \quad F_{[n]}^i E = F^{in} E, \quad F_m^i E = F^{i+m} E, \quad i \in \mathbb{Z}.$$

Then all rescalings and shifts of a given filtration are mutually commensurable. For any object  $E$  equipped with a filtration  $F^\bullet$ , and for any integer  $n \geq 1$ , we will denote by  $E^{[n]}$  the filtered object  $\langle E, F_{[n]}^\bullet \rangle$ .

**1.2 Filtered complexes.** A *filtered complex* in  $\mathcal{C}$  is a filtered object in the abelian category  $C_\bullet(\mathcal{C})$  of unbounded chain complexes in  $\mathcal{C}$ . We denote the category of filtered complexes by  $CF_\bullet(\mathcal{C})$ . A map  $f : E_\bullet \rightarrow E'_\bullet$  in  $CF_\bullet(\mathcal{C})$  is a *filtered quasiisomorphism* if the corresponding map  $\mathrm{gr}^i(f) : \mathrm{gr}^i E_\bullet \rightarrow \mathrm{gr}^i E'_\bullet$  is



a quasiisomorphism for any integer  $i$ . Just like inverting quasiisomorphisms in  $C_*(\mathcal{C})$  gives the derived category  $\mathcal{D}(\mathcal{C})$ , inverting filtered quasiisomorphisms in  $CF_*(\mathcal{C})$  gives the triangulated *filtered derived category*  $\mathcal{DF}(\mathcal{C})$ , see e.g. [BBD]. For any filtered quasiisomorphism  $f : \langle E_\bullet, F^\bullet \rangle \rightarrow \langle E'_\bullet, F'^\bullet \rangle$  and any integers  $i \leq j$ , the induced map

$$F^{[i,j]}(f) : F^{[i,j]}E_\bullet \rightarrow F^{[i,j]}E'_\bullet$$

is also a quasiisomorphism, so that for any object  $E \in \mathcal{DF}(\mathcal{C})$ , we have well-defined objects  $F^{[i,j]}E \in \mathcal{D}(\mathcal{C})$  and commutative squares (1.1). On the other hand, the map  $f : E_\bullet \rightarrow E'_\bullet$  is *not* required to be a quasiisomorphism, and neither are the maps  $f : F^i E_\bullet \rightarrow F^i E'_\bullet$ . Thus for an arbitrary filtered complex  $\langle E_\bullet, F^\bullet \rangle$ , the object  $E \in \mathcal{D}(\mathcal{C})$  corresponding to  $E_\bullet$  cannot in general be recovered from the object in  $\mathcal{DF}(\mathcal{C})$  corresponding to  $\langle E_\bullet, F^\bullet \rangle$ . However, every object  $E \in \mathcal{DF}(\mathcal{C})$  can be represented by a complex  $E_\bullet$  in  $\mathcal{C}$  equipped with a convergent filtration  $F^\bullet$ , and such a representative is unique up to a quasiisomorphism. Thus one can think of objects in  $\mathcal{DF}(\mathcal{C})$  as corresponding to convergent filtered complexes. In particular, for any  $E \in \mathcal{DF}(\mathcal{C})$ , we have a well-defined completion  $\widehat{E} \in \mathcal{D}(\mathcal{C})$  (explicitly, it is still given by (1.2), where the limits have to be replaced with their derived functors). Sending  $E$  to  $\widehat{E}$  gives a completion functor

$$(1.5) \quad \mathcal{DF}(\mathcal{C}) \rightarrow \mathcal{D}(\mathcal{C}).$$

The filtration  $F^\bullet$  on  $E$  generates the standard spectral sequence

$$(1.6) \quad H_*(\mathrm{gr}^\bullet(E)) \Rightarrow H_*(\widehat{E}),$$

where  $H_*(-)$  stands for homology objects, and the spectral sequence converges in the usual sense: its  $E_\infty$ -term is naturally identified with the associated graded quotient of the right-hand side with respect to the natural filtration.

The two most standard examples of a filtration on a complex are the following ones.

**Example 1.4.** For any complex  $E_\bullet$  in  $\mathcal{C}$ , the *stupid filtration*  $F^\bullet E_\bullet$  and the *canonical filtration*  $\tau^\bullet E_\bullet$  are given by

$$F^i E_j = \begin{cases} E_j, & j + i \leq 0, \\ 0, & j + i > 0, \end{cases}, \quad \tau^i E_j = \begin{cases} 0, & j > i, \\ \mathrm{Ker} d, & j = i, \\ E^j, & j < i. \end{cases}$$

Both filtrations are convergent.

**Remark 1.5.** Our  $\tau^i$  is usually denoted  $\tau_{\leq -i}$ ; we change notation to keep all filtrations decreasing and simplify the statements.

Canonical filtration is called canonical because it is compatible with quasiisomorphisms: any quasiisomorphism  $f : E_\bullet \rightarrow E'_\bullet$  in  $C_\bullet(\mathcal{C})$  is a filtered quasiisomorphism with respect to the canonical filtration (the spectral sequence (1.6) trivially degenerates for dimension reasons). Therefore sending  $E_\bullet$  to  $\langle E_\bullet, \tau^\bullet \rangle$  descends to a functor  $\mathcal{D}(\mathcal{C}) \rightarrow \mathcal{DF}(\mathcal{C})$ . This functor is fully faithful. For any integer  $i$ , the associated graded quotient  $\mathrm{gr}_\tau^i E_\bullet$  with respect to the canonical filtration is naturally quasiisomorphic to the  $i$ -th homology object  $H_i(E_\bullet)$  of the complex  $E_\bullet$ . To make this quasiisomorphism more explicit, one introduces another functorial decreasing filtration  $\beta^\bullet$  on  $E_\bullet$  given by

$$(1.7) \quad \beta^i E_j = \begin{cases} 0, & j > i, \\ \mathrm{Im} \, d, & j = i, \\ E_j, & j < i. \end{cases}$$

Then  $\tau^{i+1} E_\bullet \subset \beta^i E_\bullet \subset \tau^i E_\bullet$ , the embedding  $\tau^{i+1} E_\bullet \rightarrow \beta^i E_\bullet$  is a quasiisomorphism, and the quotient  $\tau^i E_\bullet / \beta^i E_\bullet$  is isomorphic to the homology object  $H_i(E_\bullet)$  placed in homological degree  $i$ .

Stupid filtration is sufficiently functorial to define a functor  $C_\bullet(\mathcal{C}) \rightarrow \mathcal{DF}(\mathcal{C})$ ,  $E_\bullet \mapsto \langle E_\bullet, F^\bullet \rangle$ , but the functor does not factor through  $\mathcal{D}(\mathcal{C})$ : conversely, it is already a fully faithful embedding on the level of the category  $C_\bullet(\mathcal{C})$ , and it identifies  $C_\bullet(\mathcal{C})$  with the heart of a natural  $t$ -structure on  $\mathcal{DF}(\mathcal{C})$  (see e.g. [Be1, Appendix]).

For a more interesting example of a filtered complex, denote by  $C_{\bullet,\bullet}(\mathcal{C})$  the category of chain bicomplexes in  $\mathcal{C}$ , again unbounded in either coordinate. We will call the first coordinate the *vertical direction*, and the second coordinate the *horizontal direction*. Then any  $E_{\bullet,\bullet} \in C_{\bullet,\bullet}(\mathcal{C})$  has at least four natural filtrations – namely, the stupid and the canonical filtration in either of the two directions. We also have the two totalization functors from  $C_{\bullet,\bullet}(\mathcal{C})$  to  $C_\bullet(\mathcal{C})$ , namely, the sum-total and the product-total complex functors

$$\mathrm{tot}, \mathrm{Tot} : C_{\bullet,\bullet}(\mathcal{C}) \rightarrow C_\bullet(\mathcal{C}),$$

with a natural map  $\mathrm{tot} \rightarrow \mathrm{Tot}$  between them. The four filtrations on  $E_{\bullet,\bullet}$  induce filtrations on  $\mathrm{tot}(E_{\bullet,\bullet})$  and  $\mathrm{Tot}(E_{\bullet,\bullet})$ . Sometimes, some of these filtrations are automatically convergent. For example, assume that  $E_{\bullet,\bullet}$  is a second-quadrant bicomplex – that is,  $E_{i,j} = 0$  unless  $i \leq 0 \leq j$ . Then

$\text{Tot}(E_{\bullet,\bullet})$  is convergent with respect to the stupid filtration in the vertical direction and the canonical filtration in the horizontal direction, while  $\text{tot}(E_{\bullet,\bullet})$  is convergent with respect to the stupid filtration in the horizontal direction and the canonical filtration in the vertical direction. Thus the spectral sequences (1.6) for  $\text{tot}(E_{\bullet,\bullet})$  and  $\text{Tot}(E_{\bullet,\bullet})$  are different, and the natural map  $\text{tot}(E_{\bullet,\bullet}) \rightarrow \text{Tot}(E_{\bullet,\bullet})$  need not be a quasiisomorphism. Here is the standard example of such a situation.

**Lemma 1.6.** *Assume given objects  $E_n \in \mathcal{C}$  and morphisms  $e_n : E_n \rightarrow E_{n+1}$ ,  $n \geq 0$ , and consider the bicomplex  $E_{\bullet,\bullet}$  given by  $E_{-n,n} = E_{-(n+1),n} = E_n$ ,  $E_{i,j} = 0$  otherwise, with the non-trivial horizontal differentials given by the identity maps, and the vertical differentials given by the maps  $e_i$ . Assume that either the category  $\mathcal{C}$  satisfies AB5 (filtered direct limits are exact), or all the maps  $e_i$  are injective. Then  $\text{Tot}(E_{\bullet,\bullet})$  is quasiisomorphic to  $E_0$  placed in degree 0, while  $\text{tot}(E_{\bullet,\bullet})$  is quasiisomorphic to the cone of the natural map*

$$E_0 \rightarrow \lim_{\substack{\rightarrow \\ i}} E_i,$$

where the limit is taken with respect to the maps  $e_i$ . The natural map from  $\text{tot}(E_{\bullet,\bullet})$  to  $\text{Tot}(E_{\bullet,\bullet})$  fits into a distinguished triangle

$$\text{tot}(E_{\bullet,\bullet}) \longrightarrow \text{Tot}(E_{\bullet,\bullet}) \longrightarrow \lim_{\substack{\rightarrow \\ i}} E_i \longrightarrow .$$

**Remark 1.7.** Explicitly, the bicomplex  $E_{\bullet,\bullet}$  is given by the snake diagram

$$\begin{array}{ccccc} \dots & \xrightarrow{\text{id}} & E_3 & & \\ & \uparrow e_2 & & & \\ & E_2 & \xrightarrow{\text{id}} & E_2 & \\ & & \uparrow e_1 & & \\ & & E_1 & \xrightarrow{\text{id}} & E_1 \\ & & & \uparrow e_0 & \\ & & & E_0 & . \end{array}$$

*Proof.* The first claim is clear from (1.6). For second, compute the limit by the telescope construction. The last claim is clear.  $\square$

**1.3 Truncations and functoriality.** For a general bicomplex  $E_{\bullet,\bullet}$ , neither  $\text{tot}(E_{\bullet,\bullet})$  nor  $\text{Tot}(E_{\bullet,\bullet})$  is convergent with respect to either of the filtrations – the situation for  $\text{tot}(E_{\bullet,\bullet})$  is described in Example 1.2, and  $\text{Tot}(E_{\bullet,\bullet})$  is dual. However, the completion of  $\text{tot}(E_{\bullet,\bullet})$  with respect to the stupid filtration in the vertical direction is isomorphic to its completion with respect to the canonical filtration in the horizontal direction. We will need a slight generalization of this fact.

First of all, fix the identification  $C_{\bullet,\bullet}(\mathcal{C}) \cong C_{\bullet}(C_{\bullet}(\mathcal{C}))$  so that the outer index in  $C_{\bullet}(C_{\bullet}(\mathcal{C}))$  corresponds to the horizontal direction in  $C_{\bullet,\bullet}(\mathcal{C})$ . Since  $C_{\bullet}(\mathcal{C})$  is an abelian category in its own right, this defines the truncations functors  $\tau^{\bullet}$  and their counterparts  $\beta^{\bullet}$  of (1.7) on  $C_{\bullet,\bullet}(\mathcal{C}) \cong C_{\bullet}(C_{\bullet}(\mathcal{C}))$ . Now denote by

$$\text{tot}^f : C_{\bullet,\bullet}(\mathcal{C}) \rightarrow CF_{\bullet}(\mathcal{C})$$

the functor sending a bicomplex  $E_{\bullet,\bullet}$  to  $\text{tot}(E_{\bullet,\bullet})$  with the filtration induced by the stupid filtration in the vertical direction on  $E_{\bullet,\bullet}$ . Say that a map  $f : E_{\bullet,\bullet} \rightarrow E'_{\bullet,\bullet}$  is a *horizontal quasiisomorphism* if  $f : E_{i,\bullet} \rightarrow E'_{i,\bullet}$  is a quasiisomorphism for any integer  $i$ . Then inverting horizontal quasiisomorphisms in  $C_{\bullet,\bullet}(\mathcal{C}) \cong C_{\bullet}(C_{\bullet}(\mathcal{C}))$  gives the derived category  $\mathcal{D}(C_{\bullet}(\mathcal{C}))$ , and  $\text{tot}^f$  descends to a functor

$$\mathcal{D}(C_{\bullet}(\mathcal{C})) \rightarrow \mathcal{D}F(\mathcal{C}).$$

It is well-known (see e.g. [Be1, Appendix]) that this functor is an equivalence of categories. Thus in particular, the truncation functors  $\tau^{\bullet}$  of Example 1.4 are well-defined on the filtered derived category  $\mathcal{D}F(\mathcal{C})$ .

It turns out that in fact both  $\tau^{\bullet}$  and the related truncation functors  $\beta^{\bullet}$  of (1.7) can be defined already on the category  $CF_{\bullet}(\mathcal{C})$ . Namely, for any filtered complex  $\langle E_{\bullet}, F^{\bullet} \rangle$  with differential  $d$ , and any integers  $n, i$ , let

$$(1.8) \quad \begin{aligned} \tau^n E_i &= d^{-1}(F^{n+1-i} E_{i-1}) \cap F^{n-i} E_i \subset E_i, \\ \beta^n E_i &= F^{n+1-i} E_i + d(F^{n-i} E_{i+1}) \subset E_i. \end{aligned}$$

Then for any  $n$ ,  $\tau^n E_{\bullet}$  and  $\beta^n E_{\bullet}$  are subcomplexes in  $E_{\bullet}$ , and we have  $\tau^{n+1} E_{\bullet} \subset \beta^n E_{\bullet} \subset \tau^n E_{\bullet}$ . Equip  $\tau^n E_{\bullet}$  and  $\beta^n E_{\bullet}$  with the filtrations induced by  $F^{\bullet}$ , and denote

$$(1.9) \quad H_n(E_{\bullet}) = \tau^n E_{\bullet} / \beta^n E_{\bullet} \subset CF_{\bullet}(\mathcal{C}),$$

again equipped with the induced filtration. For any bicomplex  $E_{\bullet,\bullet}$  and any integer  $n$ , we then obviously have

$$\tau^n \text{tot}^f(E_{\bullet,\bullet}) \cong \text{tot}^f(\tau^n E_{\bullet,\bullet}), \quad \beta^n \text{tot}^f(E_{\bullet,\bullet}) \cong \text{tot}^f(\beta^n E_{\bullet,\bullet}),$$

and  $H_n(\text{tot}^f(E_{\bullet,\bullet}))$  coincides with  $\text{tot}^f(H_n(E_{\bullet,\bullet}))$ , where  $H_n(E_{\bullet,\bullet}) \in C_{\bullet}(\mathcal{C})$  is the  $n$ -th homology object of  $E_{\bullet,\bullet} \in C_{\bullet,\bullet}(\mathcal{C}) \cong C_{\bullet}(C_{\bullet}(\mathcal{C}))$ .

**Lemma 1.8.** *For any filtered complex  $\langle E_{\bullet}, F^{\bullet} \rangle$  in  $\mathcal{C}$  and any integer  $n, i$ , we have*

$$\text{gr}_F^i \tau^n E_{\bullet} \cong \tau^{n-i} \text{gr}_F^i E_{\bullet}, \quad \text{gr}_F^i \beta^n E_{\bullet} \cong \beta^{n-i} \text{gr}_F^i E_{\bullet}.$$

Moreover, the embedding  $\tau^{n+1} E_{\bullet} \rightarrow \beta^n E_{\bullet}$  is a filtered quasiisomorphism, and we have  $\text{gr}_F^i H_n(E_{\bullet}) \cong H_{n-i}(\text{gr}_F^i E_{\bullet})[n]$ , so that the filtration  $F^{\bullet}$  on  $H_n(E_{\bullet})$  is the stupid filtration shifted by  $n$ , as in (1.4). Finally, for any integer  $i$ , the filtrations  $\tau^{\bullet}$ ,  $\beta^{\bullet}$  and  $F^{\bullet}$  on  $E_i$  are commensurable.

*Proof.* Clear. □

We will also need one fact about functoriality properties of the filtrations  $\tau^{\bullet}$ ,  $\beta^{\bullet}$ . Assume given abelian categories  $\mathcal{C}, \mathcal{C}'$ . Any additive functor  $\varphi : \mathcal{C} \rightarrow \mathcal{C}'$  naturally extends to a functor  $\varphi_{\bullet} : \mathcal{C} \rightarrow \mathcal{C}'$  by applying it termwise, that is, setting  $\varphi(E_{\bullet})_i = \varphi(E_i)$  for any complex  $E_{\bullet}$  in  $\mathcal{C}$  and any integer  $i$ . For any additive functor  $\varphi_{\bullet} : \mathcal{C} \rightarrow C_{\bullet}(\mathcal{C}')$ , we define its extension  $\varphi_{\bullet} : C_{\bullet}(\mathcal{C}) \rightarrow C_{\bullet}(\mathcal{C}')$  as the composition

$$C_{\bullet}(\mathcal{C}) \xrightarrow{\varphi_{\bullet,\bullet}} C_{\bullet}(C_{\bullet}(\mathcal{C}')) \cong C_{\bullet,\bullet}(\mathcal{C}') \xrightarrow{\text{tot}} C_{\bullet}(\mathcal{C}').$$

That is, we apply  $\varphi_{\bullet}$  termwise, and then take the sum-total complex  $\text{tot}(-)$ . Explicitly, for any complex  $E_{\bullet}$  in  $\mathcal{C}$ , the complex  $\varphi_{\bullet}(E_{\bullet})$  in  $\mathcal{C}'$  has terms

$$(1.10) \quad \varphi_{\bullet}(E_{\bullet})_i = \bigoplus_j \varphi_{i-j}(E_j).$$

We also define the functor

$$(1.11) \quad \varphi^f = \text{tot}^f \circ \varphi_{\bullet,\bullet} : C_{\bullet}(\mathcal{C}) \rightarrow CF_{\bullet}(\mathcal{C}').$$

Moreover, say that a filtration  $F^{\bullet}$  on a complex  $E_{\bullet}$  is *termwise-split* if for any  $i, j$ , the embedding  $F^i E_j \rightarrow E_j$  is a split injection, and denote by  $\overline{CF}_{\bullet}(\mathcal{C}) \subset CF_{\bullet}(\mathcal{C})$  the full subcategory spanned by complexes with termwise-split filtrations. Then since any functor sends split injections to split injections, we can further extend the functor  $\varphi^f$  to a functor

$$(1.12) \quad \varphi^f : \overline{CF}_{\bullet}(\mathcal{C}) \rightarrow \overline{CF}_{\bullet}(\mathcal{C}')$$

by setting

$$F^n \varphi^f(E_{\bullet})_i = \bigoplus_j \varphi_{i-j}(F^{n+j} E_j) \subset \varphi_{\bullet}(E_{\bullet})_i,$$

where we recall that  $\varphi_\bullet(E_\bullet)_i$  is explicitly given by (1.10). Note that if a complex  $E_\bullet$  is placed in filtered degree 0, then the filtration is tautologically termwise split, and (1.12) coincides with (1.11). If moreover  $E_\bullet$  is concentrated in homological degree 0, then  $\varphi^f(E_\bullet)$  is just  $\varphi_\bullet(E^0)$  equipped with the stupid filtration.

**Lemma 1.9.** *Assume given an additive functor  $\varphi_\bullet : \mathcal{C} \rightarrow \mathcal{C}_\bullet(\mathcal{C}')$ . Then for any integer  $n$  and any complex  $E_\bullet$  in  $\mathcal{C}$  equipped with a termwise-split filtration  $F^\bullet$ , the filtrations on the truncations  $\tau^n E_\bullet, \beta^n E_\bullet \subset E_\bullet$  of (1.8) are termwise-split, and the natural maps  $\varphi^f(\tau^n E_\bullet) \rightarrow \varphi^f(E_\bullet)$ ,  $\varphi^f(\beta^n E_\bullet) \rightarrow \varphi^f(E_\bullet)$  factor through natural maps*

$$\varphi^f(\tau^n E_\bullet) \rightarrow \tau^n(\varphi^f(E_\bullet)) \subset \varphi^f(E_\bullet), \quad \varphi^f(\beta^n E_\bullet) \rightarrow \beta^n(\varphi^f(E_\bullet)) \subset \varphi^f(E_\bullet).$$

*Moreover, if the functor  $\varphi_\bullet$  is right-exact, the map  $\varphi^f(\tau^n E_\bullet) \rightarrow \tau^n \varphi^f(E_\bullet)$  is surjective.*

*Proof.* By Lemma 1.8, for any integers  $i, n$ , we have  $(\tau^n E_\bullet)_i = F^{n-i}(\tau^n E_\bullet)_i$ ,  $F^{n+1-i}(\tau^n E_\bullet)_i = F^{n+1-i}E_i$ , and a splitting of the projection

$$F^{n-i}(\tau^n E_\bullet)_i \rightarrow \mathrm{gr}_F^{n-i}(\tau^n E_\bullet)_i \cong (\tau^i \mathrm{gr}_F^{n-i} E_\bullet)_i$$

is induced by a splitting of the projection  $F^{n-i}E_i \rightarrow \mathrm{gr}_F^{n-i}E_i$ . Therefore the filtration on  $\tau^n E_\bullet$  induced by  $F^\bullet$  is termwise-split. The same argument shows that the filtration on  $\beta^n E_\bullet$  is also termwise-split. Then applying the additive functor  $\varphi$  commutes with taking associated graded quotients with respect to  $F^\bullet$ , and by Lemma 1.8, it suffices to prove the remaining claims when  $E_\bullet$  is concentrated in a single filtered degree. Then up to a shift,  $\tau^\bullet$  is the canonical filtration,  $\beta^\bullet$  is given by (1.7), and the claims are obvious.  $\square$

## 2 Mixed complexes.

**2.1 Mixed complexes and expansions.** As in Section 1, we fix an abelian category  $\mathcal{C}$  satisfying  $AB4$  and  $AB4^*$ .

**Definition 2.1.** A *mixed complex*  $\langle V_\bullet, B \rangle$  in  $\mathcal{C}$  is a complex  $V_\bullet$  equipped with a map  $B : V_\bullet \rightarrow V_\bullet[-1]$  such that  $B^2 = 0$ . A *map of mixed complexes*  $f : \langle V_\bullet, B \rangle \rightarrow \langle V'_\bullet, B \rangle$  is a map of complexes  $f : V_\bullet \rightarrow V'_\bullet$  that commutes with  $B$ .

For every map of mixed complexes, its cone is also a mixed complex. For every integer  $n$ , the canonical truncation  $\tau^n V_\bullet$  of a mixed complex  $\langle V_\bullet, B \rangle$  is a mixed complex, and the natural embedding  $\tau^n V_\bullet \rightarrow V_\bullet$  is a map of mixed complexes.

**Definition 2.2.** The *expansion*  $\text{Exp}(\langle V_\bullet, B \rangle)$  of a mixed complex  $\langle V_\bullet, B \rangle$  is the complex isomorphic to  $V_\bullet[u^{-1}]$  as a graded object in  $\mathcal{C}$ , with  $u$  being a formal variable of cohomological degree 2, and equipped with the differential  $d + Bu$ , where  $d : V_\bullet \rightarrow V_{\bullet-1}$  is the differential in the complex  $V_\bullet$ .

We will write  $\text{Exp}(V_\bullet)$  when  $B$  is clear from the definition. The meaning of the shorthand notation  $V_\bullet[u^{-1}]$  is the same as in Example 1.2; as usual, polynomials in  $u^{-1}$  are understood as a module over polynomials in  $u$  via the identification  $k[u^{-1}] = k[u, u^{-1}]/uk[u]$ . Equivalently,  $\text{Exp}(V_\bullet)$  is the sum-total complex of the bicomplex

$$(2.1) \quad \xrightarrow{B} V_\bullet[3] \xrightarrow{B} V_\bullet[2] \xrightarrow{B} V_\bullet[1] \xrightarrow{B} V_\bullet,$$

and  $u : \text{Exp}(V_\bullet) \rightarrow \text{Exp}(V_\bullet)[2]$  is induced by the obvious map shifting the bicomplex by 1 in either direction. By definition, we have a short exact sequence of complexes

$$0 \longrightarrow V_\bullet \longrightarrow \text{Exp}(V_\bullet) \xrightarrow{u} \text{Exp}(V_\bullet)[2] \longrightarrow 0.$$

The same shorthand notation is used in the following definition.

**Definition 2.3.** The *periodic* resp. *co-periodic* resp. *polynomial periodic* expansions  $\text{Per}(V_\bullet)$  resp.  $\overline{\text{Per}}(V_\bullet)$  resp.  $\text{per}(V_\bullet)$  of a mixed complex  $\langle V_\bullet, B \rangle$  are the complexes in  $\mathcal{C}$  given by

$$\text{Per}(V_\bullet) = V_\bullet((u)), \quad \overline{\text{Per}}(V_\bullet) = V_\bullet((u^{-1})), \quad \text{per}(V_\bullet) = V_\bullet[u, u^{-1}]$$

as graded objects, each equipped with the differential  $d + Bu$ .

By definition, for every mixed complex  $\langle V_\bullet, B \rangle$ , we have natural maps

$$(2.2) \quad \text{Per}(V_\bullet) \xleftarrow{l} \text{per}(V_\bullet) \xrightarrow{r} \overline{\text{Per}}(V_\bullet).$$

In general, neither of these maps is an isomorphism, or even a quasiisomorphism. We also have

$$(2.3) \quad \text{Per}(V_\bullet) = \varprojlim^u \text{Exp}(V_\bullet).$$

$$(2.4) \quad \begin{aligned} F^i \operatorname{Per}(V_\bullet) &= \operatorname{Per}(F^i V_\bullet), \\ F^i \overline{\operatorname{Per}}(V_\bullet) &= \overline{\operatorname{Per}}(F^i V_\bullet), \\ F^i \operatorname{per}(V_\bullet) &= \operatorname{per}(F^i V_\bullet) \end{aligned}$$

for any integer  $i$ . The corresponding associated graded quotients are given by

again for any integer  $i$ .

**Definition 2.5.** A mixed complex  $\langle V_\bullet, B \rangle$  is *bounded from below* if for some integer  $n$ ,  $\tau^n V_\bullet$  is acyclic, and it is *strongly bounded from above* if for some integer  $n$ , the map  $\tau^n V_\bullet \rightarrow V_\bullet$  is a strong quasiisomorphism.

*Proof.* For any mixed complex  $\langle V_\bullet, B \rangle$ , let  $\text{per}_{\bullet,\bullet}(V_\bullet)$  be the bicomplex

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a periodic version of the bicomplex (2.1). Then by definition, we have  $\text{per}(V_\bullet) = \text{tot}(\text{per}_{\bullet,\bullet}(V_\bullet))$ , and  $\text{Per}(V_\bullet)$  is its completion with respect to the stupid filtration in the horizontal direction. Applying the spectral sequence (1.6), we get the first claim. For the second claim, it suffices to check that for a strongly acyclic mixed complex  $V_\bullet$ ,  $\text{per}(V_\bullet)$  and  $\overline{\text{Per}}(V_\bullet)$  are acyclic. This is obvious:  $\text{per}(-)$  and  $\overline{\text{Per}}(-)$  are exact functors, and they send contractible mixed complexes to contractible complexes.  $\square$

**Corollary 2.7.** *Assume given a mixed complex  $\langle V_\bullet, B \rangle$ .*

- (i) *If  $V_\bullet$  is bounded from below in the sense of Definition 2.5, then the natural map  $l : \text{per}(V_\bullet) \rightarrow \text{Per}(V_\bullet)$  is a quasiisomorphism.*
- (ii) *If  $V_\bullet$  is strongly bounded from above in the sense of Definition 2.5, then the natural map  $r : \text{per}(V_\bullet) \rightarrow \overline{\text{Per}}(V_\bullet)$  is a quasiisomorphism.*

*Proof.* By Lemma 2.6, it suffices to prove that if we have an integer  $n$  such that  $V_i = 0$  for  $i \geq n$  resp.  $i \leq n$ , then  $l$  resp.  $r$  is quasiisomorphism. In fact, under these assumptions it is even an isomorphism.  $\square$

**2.2 Cyclic groups.** To obtain a useful example of a mixed complex, fix an integer  $n \geq 1$ , and let  $C = \mathbb{Z}/n\mathbb{Z}$  be the cyclic group of order  $n$ , with the generator  $\sigma \in C$ . If we denote  $\mathbb{K}_1 = \mathbb{K}_0 = \mathbb{Z}[C]$ , then we have a natural exact sequence

$$(2.7) \quad 0 \longrightarrow \mathbb{Z} \xrightarrow{\kappa_1} \mathbb{K}_1 \xrightarrow{\text{id}-\sigma} \mathbb{K}_0 \xrightarrow{\kappa_0} \mathbb{Z} \longrightarrow 0$$

of  $\mathbb{Z}[C]$ -modules, where the action of  $C$  on  $\mathbb{Z}$  is trivial. We thus have a complex  $\mathbb{K}_\bullet$  of  $\mathbb{Z}[C]$ -modules of length 2, with homology in degrees 0 and 1 identified with  $\mathbb{Z}$ . Denote

$$B = \kappa_1 \circ \kappa_0 : \mathbb{K}_0 \rightarrow \mathbb{K}_1.$$

Then  $B \circ (\text{id} - \sigma) = (\text{id} - \sigma) \circ B = 0$ , so that  $B$  defines a map of complexes

$$(2.8) \quad B : \mathbb{K}_\bullet \rightarrow \mathbb{K}_\bullet[-1]$$

turning  $\mathbb{K}_\bullet$  into a mixed complex. Explicitly, we have

$$B = \text{id} + \sigma + \cdots + \sigma^{n-1}.$$

Moreover, fix a ring  $R$ . Then the category  $\mathcal{C}$  of left  $R[C]$ -modules is an abelian category satisfying  $AB4$  and  $AB4^*$ , and for any  $R[C]$ -module  $E$ , we have a natural mixed complex

$$(2.9) \quad \mathbb{K}_\bullet(E) = \mathbb{K}_\bullet \otimes E$$

in  $\mathcal{C}$ , with the differential  $B$  induced by the map (2.8). The expansion  $\text{Exp}(\mathbb{K}_\bullet(E))$  is then the standard periodic free resolution of  $E$ . Since the complex  $\mathbb{K}_\bullet(E)$  is concentrated in a finite number of degrees, the maps (2.2) for  $\mathbb{K}_\bullet(E)$  are quasiisomorphisms – in fact, isomorphisms – and the complex  $\text{per}(\mathbb{K}_\bullet(E)) \cong \text{Per}(\mathbb{K}_\bullet(E)) \cong \overline{\text{Per}}(\mathbb{K}_\bullet(E))$  is acyclic.

Taking coinvariants with respect to  $C$ , – or equivalently, with respect to the generator  $\sigma \in C$  – we obtain a mixed complex  $\mathbb{K}_\bullet(E)_\sigma$  in the category of  $R$ -modules. Its expansion

$$(2.10) \quad C_\bullet(C, E) = \text{Exp}(\mathbb{K}_\bullet(E)_\sigma) \cong \text{Exp}(\mathbb{K}_\bullet(E))_\sigma$$

is the standard periodic complex computing the homology groups  $H_\bullet(C, E)$ , and the periodic expansion

$$(2.11) \quad \check{C}_\bullet(C, E) = \text{Per}(\mathbb{K}_\bullet(E)_\sigma) \cong \text{Per}(\mathbb{K}_\bullet(E))_\sigma$$

computes the Tate homology groups  $\check{H}_\bullet(C, E)$ . All the non-trivial terms of these complexes are isomorphic to  $E$  as an  $R$ -modules, and the differentials are  $\text{id} - \sigma$  resp.  $\text{id} + \sigma + \cdots + \sigma^{n-1}$  in odd resp. even degrees.

More generally, let  $E_\bullet$  be a complex of  $R[C]$ -modules. Then  $\mathbb{K}_\bullet(E_\bullet) = \mathbb{K}_\bullet \otimes E_\bullet$  equipped with a map (2.8) is still a mixed complex of  $R[C]$ -modules, and taking coinvariants, we obtain a mixed complex  $\mathbb{K}_\bullet(E_\bullet)_\sigma$  of  $R$ -modules. Since  $\mathbb{K}_\bullet(E_\bullet)_\sigma$  is no longer concentrated in a finite range of degrees, the maps (2.2) need not be quasiisomorphisms anymore. However, we can still define the Tate homology complex  $\check{C}_\bullet(C, E_\bullet)$  by (2.11), and for any quasiisomorphism  $E_\bullet \rightarrow E'_\bullet$  of complexes of  $R[C]$ -modules, the corresponding map

$$\check{C}_\bullet(C, E_\bullet) \rightarrow \check{C}_\bullet(C, E'_\bullet)$$

is a quasiisomorphism by Lemma 2.6. Taking other periodic expansions of Definition 2.3, we obtain two more versions of the Tate cohomology complex that we denote by

$$(2.12) \quad \check{\check{C}}_\bullet(C, E_\bullet) = \overline{\text{Per}}(\mathbb{K}_\bullet(E_\bullet)_\sigma), \quad \tilde{C}_\bullet(C, E_\bullet) = \text{per}(\mathbb{K}_\bullet(E_\bullet)_\sigma).$$

These are not in general invariant under quasiisomorphisms. To get an invariance result, we proceed as in Definition 2.4.

**Definition 2.8.** A complex  $E_\bullet$  in an abelian category  $\mathcal{C}$  is *strongly acyclic* if it is a finite extension of contractible complexes. A map  $f : E_\bullet \rightarrow E'_\bullet$  is a *strong quasiisomorphism* if its cone is strongly acyclic.

**Definition 2.9.** A complex  $E_\bullet$  in an abelian category  $\mathcal{C}$  is *bounded from below* resp. *strongly bounded from below* if for some integer  $n$ ,  $\tau^n E_\bullet$  is acyclic resp. strongly acyclic, and it is *bounded from above* resp. *strongly bounded from above* if for some integer  $n$ , the map  $\tau^n E_\bullet \rightarrow E_\bullet$  is a quasiisomorphism resp. strong quasiisomorphism.

**Lemma 2.10.** (i) *For every strong quasiisomorphism  $f$  of complexes of  $R[C]$ -modules, the maps  $\text{per}(f)$  and  $\overline{\text{Per}}(f)$  are quasiisomorphisms.*

(ii) *If a complex  $E_\bullet$  of  $R[C]$ -modules is bounded from below in the sense of Definition 2.9, then the map  $l : \tilde{C}_\bullet(C, E_\bullet) \rightarrow \check{C}_\bullet(C, E_\bullet)$  is a quasiisomorphism. If  $E_\bullet$  is strongly bounded from above, then the map  $r : \tilde{C}_\bullet(C, E_\bullet) \rightarrow \check{C}_\bullet(C, E_\bullet)$  is a quasiisomorphism.*

*Proof.* Since  $\mathbb{K}(E)_\sigma$  is an exact functor, the claims immediately follow from Lemma 2.6 resp. Corollary 2.7.  $\square$

**Lemma 2.11.** *For any complex  $E_\bullet$  of  $R[C]$ -modules, equip the Tate complex  $\tilde{C}_\bullet(C, E_\bullet)$  with the filtration induced by the stupid resp. canonical filtration on  $E_\bullet$ . Then its completion coincides with the complex  $\check{C}_\bullet(C, E_\bullet)$  resp.  $\check{C}_\bullet(C, E_\bullet)$ .*

*Proof.* Clear.  $\square$

We will also need a version of homology complexes twisted by a sign. Namely, let  $\sigma^\dagger = (-1)^{|C|+1}\sigma \in R[C]$ , where  $|C|$  is the order of the cyclic group  $C$ . Then  $\sigma^\dagger$  is an invertible element of order  $|C|$ , so that any  $R[C]$ -module  $E$  gives another  $R[C]$ -module  $E^\dagger$  with the generator of  $C$  acting by  $\sigma^\dagger$ . Alternatively,  $E^\dagger = E \otimes I$ , where  $I$  is the one-dimensional sign representation:  $I = \mathbb{Z}$  as a  $\mathbb{Z}$ -module, and  $\sigma$  acts by  $(-1)^{|C|+1}$ . Then taking coinvariants with respect to  $\sigma^\dagger$ , we obtain a mixed complex  $\mathbb{K}(E)_{\sigma^\dagger}$  and its expansions

$$(2.13) \quad \begin{aligned} C_\bullet^\dagger(C, E) &= \text{Exp}(\mathbb{K}(E)_{\sigma^\dagger}) \cong \text{Exp}(\mathbb{K}(E))_{\sigma^\dagger}, \\ \check{C}_\bullet^\dagger(C, E) &= \text{Per}(\mathbb{K}(E)_{\sigma^\dagger}) \cong \text{Per}(\mathbb{K}(E))_{\sigma^\dagger}, \end{aligned}$$

the twisted versions of (2.10) and (2.11). Explicitly, all the terms in  $\check{C}_\bullet^\dagger(C, E)$  are isomorphic to  $E$  as  $R$ -modules, with the alternating differentials  $\text{id} - \sigma^\dagger$ ,

$\text{id} + \sigma^\dagger + \dots (\sigma^\dagger)^{n-1}$ . More generally, for any complex  $E_\bullet$  of  $R[C]$ -modules, we have the mixed complex  $\mathbb{K}_\bullet(E_\bullet)_{\sigma^\dagger}$  and its expansions

$$(2.14) \quad \begin{aligned} C_\bullet^\dagger(C, E_\bullet) &= \text{Exp}(\mathbb{K}_\bullet(E_\bullet)_{\sigma^\dagger}), & \check{C}_\bullet^\dagger(C, E_\bullet) &= \overline{\text{Per}}(\mathbb{K}_\bullet(E_\bullet)_{\sigma^\dagger}), \\ \check{\check{C}}_\bullet^\dagger(C, E_\bullet) &= \overline{\text{Per}}(\mathbb{K}_\bullet(E_\bullet)_{\sigma^\dagger}), & \tilde{C}_\bullet^\dagger(C, E_\bullet) &= \text{per}(\mathbb{K}_\bullet(E_\bullet)_{\sigma^\dagger}), \end{aligned}$$

the twisted versions of (2.10), (2.11) and (2.12). We of course have

$$C_\bullet^\dagger(C, E_\bullet) \cong C_\bullet(C, E_\bullet \otimes I),$$

and similarly for the periodic complexes, so that the general properties of periodic complexes also hold for their twisted versions. To wit,  $\check{C}_\bullet^\dagger(C, -)$  sends quasiisomorphisms to quasiisomorphisms, and we have the maps (2.2), Lemma 2.10 and Lemma 2.11.

Finally, let us note that one can modify the constructions of this Subsection in the following way. Take a larger finite cyclic group  $C'$  containing  $C$ , and denote by  $\mathbb{K}'_\bullet$  the complex (2.7) for the group  $C'$  considered as a complex of  $R[C]$ -modules by restriction with respect to the embedding  $C \subset C'$ . Then  $\mathbb{K}'_\bullet$  is also a mixed complex, and for any  $R[C]$ -module  $E$ , we can form a mixed complex  $\mathbb{K}'_\bullet(E)_\sigma = (\mathbb{K}' \otimes E)_\sigma$  and its expansions

$$(2.15) \quad C_\bullet(C, C', E) = \text{Exp}(\mathbb{K}'_\bullet(E)_\sigma), \quad \check{C}_\bullet(C, C', E) = \text{Per}(\mathbb{K}'_\bullet(E)_\sigma).$$

However, both  $\mathbb{K}_\bullet$  and  $\mathbb{K}'_\bullet$  are complexes of projective  $\mathbb{Z}[C]$ -modules, and it is easy to see that there a decomposition

$$\mathbb{K}'_\bullet \cong \mathbb{K}_\bullet \oplus K_\bullet$$

of mixed complexes of  $\mathbb{Z}[C]$ -modules with  $K_\bullet$  contractible in the sense of Definition 2.4. Therefore the complexes (2.15) are canonically chain-homotopy equivalent to their counterparts (2.10), (2.11). Analogously, for any complex  $E_\bullet$  of  $R[C]$ -modules, one can define complexes  $\check{C}_\bullet(C, C', E_\bullet)$ ,  $\check{\check{C}}_\bullet(C, C', E_\bullet)$ ,  $\tilde{C}_\bullet(C, C', E_\bullet)$  and their twisted versions as in (2.14); however, all these complexes are canonically chain-homotopy equivalent to their counterparts without the group  $C'$ .

**2.3 Mixed resolutions.** Now return to the general situation of Subsection 2.1:  $\mathcal{C}$  is again an arbitrary abelian category satisfying  $AB4$  and  $AB4^*$ . We note that while every map of mixed complexes induces a map of their expansions, the converse is not true – there are useful  $u$ -equivariant maps of expansions that do not come from maps of mixed complexes. We will need one example of this kind.

**Definition 2.12.** A *mixed resolution* of an object  $M \in \mathcal{C}$  is a mixed complex  $\langle M_\bullet, B \rangle$  in  $\mathcal{C}$ , with homology objects  $H_\bullet(M_\bullet)$ , equipped with a map  $a : M_0 \rightarrow M$  such that

- (i)  $M_i = 0$  for  $i < 0$ , and  $H_i(M_\bullet) = 0$  for  $i \geq 2$ ,
- (ii)  $a : M_0 \rightarrow M$  induces an isomorphism  $H_0(M_\bullet) \cong M$ , and
- (iii)  $B : M_\bullet \rightarrow M_\bullet[-1]$  induces an isomorphism  $H_1(M_\bullet) \cong H_0(M_\bullet)$ .

The *cohomology class*

$$\alpha \in \text{Ext}^2(M, M) \cong \text{Ext}^2(H_0(M_\bullet), H_1(M_\bullet))$$

of a mixed resolution  $M_\bullet$  is the class it represents by Yoneda as a complex.

**Lemma 2.13.** Assume given two mixed resolutions  $P_\bullet, M_\bullet$  of the same object  $M \in \mathcal{C}$  and with the same cohomology class  $\alpha \in \text{Ext}^2(M, M)$ , and assume that  $P_i$  is a projective object in  $\mathcal{C}$  for any  $i \geq 0$ . Then there exists a  $u$ -equivariant map

$$\nu : \text{Exp}(P_\bullet) \rightarrow \text{Exp}(M_\bullet)$$

that induces the given isomorphism  $H_0(P_\bullet) \cong H_0(M_\bullet) \cong M$  in degree 0.

*Proof.* Since  $P_0$  is projective, the isomorphism  $H_0(P_\bullet) \rightarrow H_0(M_\bullet)$  can be lifted to a map  $\nu_0 : P_0 \rightarrow M_0$ , and since both mixed resolutions have the same cohomology class, and  $P_i$  is projective for any  $i \geq 0$ , the map  $\nu_0$  can be extended to a map of complexes  $\nu_0 : P_\bullet \rightarrow M_\bullet$  that induces the given isomorphism  $H_1(P_\bullet) \cong H_0(P_\bullet) \cong H_0(M_\bullet) \cong H_1(M_\bullet)$  on homology in degree 1. For any mixed complex  $V_\bullet$  and any  $i \geq 0$ , denote by  $F_i \text{Exp}(V_\bullet) \subset \text{Exp}(V_\bullet)$  be  $(-i)$ -th term of the stupid filtration of the bicomplex (2.1) in the horizontal direction. Then by definition,  $u$  sends  $F_i \text{Exp}(V_\bullet)$  into  $F_{i-1} \text{Exp}(V_\bullet)$ , and we have  $\text{gr}_F^i \text{Exp}(V_\bullet) \cong V_\bullet[i]$ . By induction, we may assume that we are given a  $u$ -equivariant map of complexes

$$\nu_i : F_i \text{Exp}(P_\bullet) \rightarrow F_i \text{Exp}(M_\bullet),$$

and we need to extend it to a  $u$ -equivariant map  $\nu_{i+1}$ . First extend it to a  $u$ -equivariant graded map  $\tilde{\nu}_{i+1}$  by setting

$$\tilde{\nu}_{i+1} = u^{-1} \nu_i u$$

on  $u^{-(i+1)} P_\bullet \subset F_i \text{Exp}(P_\bullet)$ . Then while  $\tilde{\nu}_{i+1}$  does not necessarily commute with the differential  $d + uB$ , we know by induction that the commutator

$$(d + uB) \tilde{\nu}_{i+1} + \tilde{\nu}_{i+1} (d + uB) : F_{i+1} \text{Exp}(P_\bullet) \rightarrow F_{i+1} \text{Exp}(M_\bullet)$$

is divisible by  $u^i$ . That is, we have  $(d + uB)\tilde{\nu}_{i+1} + \tilde{\nu}_{i+1}(d + uB) = u^i\tau_i$  for some map  $\tau_i : P_\bullet \rightarrow M_\bullet$ . This is a map of complexes, and moreover, it is equal to 0 on homology: if  $i = 1$ , this follows from our construction of the map  $\nu_0$ , and for  $i \geq 2$ , this follows directly from Definition 2.12 (i). Since  $P_\bullet$  is a complex of projective objects,  $\tau_i$  must be homotopic to 0 – that is,  $\tau_i = dh_i + h_id$  for some graded map  $h_i : P_\bullet \rightarrow M_{\bullet+1}$ . To finish the proof, take  $\nu_i = \tilde{\nu}_i + u^ih_i$ .  $\square$

### 3 Cyclic homology.

**3.1 Cyclic complexes.** We will use the same notation and conventions as in [Ka3, Section 1]; in particular, for any small category  $I$  and ring  $R$ , we denote by  $\text{Fun}(I, R)$  the abelian category of functors from  $I$  to the category of  $R$ -modules, and we denote by  $\mathcal{D}(I, R)$  its derived category. For any functor  $\gamma : I' \rightarrow I$  between small categories, we denote by  $\gamma^* : \text{Fun}(I', R) \rightarrow \text{Fun}(I, R)$  the pullback functors, and we denote by  $\gamma_!, \gamma_* : \text{Fun}(I, R) \rightarrow \text{Fun}(I', R)$  its left and right-adjoint. If  $I'$  is the point category, and  $\gamma$  is the tautological projection, then by definition, for any  $E \in \mathcal{D}(I, R)$ ,

$$L^\bullet \gamma_! E = H_\bullet(I, E), \quad R^\bullet \gamma_* E = H^\bullet(I, E)$$

are the homology resp. cohomology groups of the category  $I$  with coefficients in  $E$ .

To study cyclic homology, we will use A. Connes' cyclic category  $\Lambda$  of [C] and its  $l$ -fold covers  $\Lambda_l$ ,  $l \geq 1$ , of [FT, Appendix, A2]. The exact definitions can be found for example in [Ka3, Section 1]. We will need to know that for any  $l \geq 1$ , objects in  $\Lambda_l$  are numbered by non-negative integers, with  $[n] \in \Lambda_l$  being the object corresponding to  $n \geq 1$ , and that we have natural functors

$$(3.1) \quad i_l, \pi_l : \Lambda_l \rightarrow \Lambda$$

given by  $i_l([n]) = [nl]$ ,  $\pi_l([n]) = [n]$  on the level of objects. If  $l = 1$ , both functors are equivalences. Objects  $[n] \in \Lambda$  can be geometrically thought of as cellular decompositions of the circle  $S^1$ ; 0-cells of such a decomposition are called *vertices*. The set of vertices corresponding to  $[n] \in \Lambda$  is denoted  $V([n])$ , and this gives a functor  $V$  from  $\Lambda$  to the category of finite sets. For any morphism  $f : [n'] \rightarrow [n]$  and any vertex  $v \in V([n])$ , the set  $f^{-1}(v) \subset V([n'])$  carries a natural total order. For any  $n, l \geq 1$ , the automorphism

group  $\text{Aut}([n])$  of the object  $[n] \in \Lambda_l$  is naturally identified with the cyclic group  $\mathbb{Z}/nl\mathbb{Z}$ . Moreover, for any  $l \geq 1$ , we have a natural functor

$$(3.2) \quad j_l : \Delta^o \rightarrow \Lambda_l,$$

where  $\Delta^o$  is the opposite to the category of finite non-empty totally ordered sets. Contrary to the standard usage, we denote by  $[n] \in \Delta$  the set with  $n$  elements, so that  $j_l([n]) = [n]$ . For  $l = 1$ , we simplify notation by writing  $j = j_1$ , and for any  $l \geq 1$ , we have  $\pi_l \circ j_l \cong j$ . The functor  $j$  is an equivalence between  $\Delta^o$  and the category of objects  $[n] \in \Lambda$  with a distinguished vertex  $v \in V([n])$ .

Recall that for any simplicial  $R$ -module  $M \in \text{Fun}(\Delta^o, R)$ , the homology  $H_*(\Delta^o, M)$  can be computed by the *standard complex*  $CH_*(M)$  with terms  $CH_i(M) = M([i+1])$ ,  $i \geq 0$ , and the differential given by

$$(3.3) \quad d = \sum_{0 \leq j \leq i} (-1)^j d_j^i,$$

where  $d_j^i : [i] \rightarrow [i+1]$  are the standard face maps. We can also consider the complex  $CH'_*(M)$  with the same terms  $CH'_i(M) = M([i+1])$  and with the differential

$$(3.4) \quad d' = \sum_{0 \leq j < i} (-1)^j d_j^i.$$

This complex is canonically contractible. More generally, if we extend the functors  $CH_*(-)$  and  $CH'_*(-)$  to complexes as in (1.10), then for any complex  $M_\bullet$  of simplicial  $R$ -modules, the complex  $CH'_*(M_\bullet)$  is acyclic, and the complex  $CH_*(M_\bullet)$  computes the homology  $H_*(\Delta^o, M_\bullet)$  of the category  $\Delta^o$  with coefficients in  $M_\bullet$ . This homology is denoted  $HH_*(M_\bullet)$ .

Now, it is well-known and easy to check that if  $M = j_l^* E$  for  $l \geq 1$  and  $E \in \text{Fun}(\Lambda_l, R)$ , then  $d \circ (\text{id} - \sigma^\dagger) = (\text{id} - \sigma^\dagger) \circ d'$ , where in every degree  $i$ ,  $\sigma$  is the generator of the cyclic group  $\mathbb{Z}/l(i+1)\mathbb{Z} \cong \text{Aut}([i+1])$ , and  $\sigma^\dagger$  is its twist by the sign, as in Subsection 2.2. Therefore we have a natural map of complexes

$$(3.5) \quad \text{id} - \sigma^\dagger : CH'_*(j_l^* E) \rightarrow CH_*(j_l^* E).$$

In particular, if we denote

$$(3.6) \quad cc_i(E) = CH_i(j_l^* E)_{\sigma^\dagger} = E([i+1])_{\sigma^\dagger} = \text{Coker}(\text{id} - \sigma^\dagger), \quad i \geq 0,$$

then the differential  $d$  of (3.3) descends to a well-defined differential  $d : cc_\bullet(E) \rightarrow cc_\bullet(E)$  turning  $cc_\bullet(E)$  into a functorial complex. We call it the *reduced cyclic complex* of the object  $E \in \text{Fun}(\Lambda_l, R)$ .

We note right away that if  $l$  is odd, then for any  $n \geq 1$ ,  $l(n+1)$  has the same parity as  $(n+1)$ , so that the sign twist in the definition of  $\sigma^\dagger$  for the objects  $[n] \in \Lambda_l$ ,  $[n] \in \Lambda$  is the same. Then for any object  $E$  in  $\text{Fun}(\Lambda_l, R)$ , (3.6) provides a functorial isomorphism

$$(3.7) \quad cc_\bullet(\pi_l E) \cong cc_\bullet(E),$$

where  $\pi_l$  is the functor (3.1).

To define the usual cyclic complex, we recall from [Ka3, Section 1] that the exact sequences (2.7) for the groups  $\text{Aut}([n])$ ,  $[n] \in \Lambda_l$  fit together into a single exact sequence

$$(3.8) \quad 0 \longrightarrow \mathbb{Z} \xrightarrow{\kappa_1} \mathbb{K}_1 \longrightarrow \mathbb{K}_0 \xrightarrow{\kappa_0} \mathbb{Z} \longrightarrow 0$$

in the category  $\text{Fun}(\Lambda_l, \mathbb{Z})$ , with  $\mathbb{Z}$  being the constant functor with value  $\mathbb{Z}$ . Then for any  $E \in \text{Fun}(\Lambda_l, R)$ , we have a natural length-2 complex  $\mathbb{K}_\bullet(E) = \mathbb{K}_\bullet \otimes E$  whose homology objects are  $E$  in degree 0 and 1. Moreover, equipping  $\mathbb{K}_\bullet(E)$  with the differential  $B = \kappa_1 \circ \kappa_0$  turns it into a mixed complex in  $\text{Fun}(\Lambda_l, R)$ . Applying the functor  $cc_\bullet$  and taking the sum-total complex, we obtain a functorial mixed complex

$$CH_\bullet(E) = \text{tot}(cc_\bullet(\mathbb{K}_\bullet(E)))$$

in the category of  $R$ -modules.

By definition,  $CH_\bullet(E)$  is the cone of a map  $cc_\bullet(\mathbb{K}_1(E)) \rightarrow cc_\bullet(\mathbb{K}_0(E))$  induced by the map  $\mathbb{K}_1 \rightarrow \mathbb{K}_0$ . Explicitly, we have natural identifications

$$(3.9) \quad cc_\bullet(\mathbb{K}_0(E)) \cong CH_\bullet(j_l^* E), \quad cc_\bullet(\mathbb{K}_1(E)) \cong CH'_\bullet(j_l^* E),$$

and the map is exactly the map (3.5). Thus we have a natural injective map

$$(3.10) \quad CH_\bullet(j_l^* E) \rightarrow CH_\bullet(E),$$

and the quotient  $CH'_\bullet(j_l^* E)$  is an acyclic complex, so that (3.10) is a quasi-isomorphism. The homology of the complex  $CH_\bullet(E)$  is denoted  $HH_\bullet(E)$ , and it coincides with  $HH_\bullet(j_l^* E)$ . The augmentation map  $\kappa_0 : \mathbb{K}_\bullet(E) \rightarrow E$  induces a natural map

$$(3.11) \quad \gamma : CH_\bullet(E) \rightarrow cc_\bullet(E).$$

In terms of the identifications (3.9), this map is given by the projections from  $CH_i(j_l^* E)$ ,  $i \geq 0$ , to their quotients  $cc_i(E)$  of (3.6).



**Definition 3.1.** For any  $l \geq 1$  and any  $E \in \text{Fun}(\Lambda_l, R)$ , the *cyclic complex*  $CC_\bullet(E)$  is given by

$$CC_\bullet(E) = \text{Exp}(CH_\bullet(E)),$$

where  $\text{Exp}$  is the expansion of Definition 2.2. The homology of the complex  $CC_\bullet(E)$  is denoted  $HC_\bullet(E)$ .

Explicitly,  $CC_\bullet(E)$  is the total complex of a bicomplex whose even-numbered columns are all isomorphic to  $CH_\bullet(j_l^* E)$ , and whose odd-numbered columns are acyclic. For  $l = 1$ ,  $CC_\bullet(E)$  is the standard cyclic complex of a cyclic object  $E \in \text{Fun}(\Lambda, R)$  (see e.g. [L, Section 2.1.2]). By definition, we have the embedding  $CH_\bullet(E) \rightarrow CC_\bullet(E)$ , and it fits into a short exact sequence of complexes

$$0 \longrightarrow CH_\bullet(E) \longrightarrow CC_\bullet(E) \xrightarrow{u} CC_\bullet(E)[2] \longrightarrow 0,$$

where  $u$  is the periodicity map of Definition 2.2. The map  $\gamma$  of (3.11) extends to a natural augmentation map

$$(3.12) \quad \alpha : CC_\bullet(E_\bullet) \rightarrow cc_\bullet(E_\bullet).$$

If for any  $[n] \in \Lambda_l$ ,  $H_i(\mathbb{Z}/nl\mathbb{Z}, E([n])) = 0$  for  $i \geq 1$ , then this map is a quasiisomorphism (this is obvious from the second spectral sequence for the bicomplex that gives  $CC_\bullet(E)$ ).

More generally, we extend the functor  $cc_\bullet(-)$  to complexes as in (1.10). Then for every complex  $E_\bullet$  in  $\text{Fun}(\Lambda_l, R)$ , we denote by  $\mathbb{K}_\bullet(E_\bullet)$  the cone of the natural map  $\mathbb{K}_1 \otimes E_\bullet \rightarrow \mathbb{K}_0 \otimes E_\bullet$ , and we denote

$$(3.13) \quad CH_\bullet(E_\bullet) = cc_\bullet(\mathbb{K}_\bullet(E_\bullet)), \quad CC_\bullet(E_\bullet) = \text{Exp}(CH_\bullet(E_\bullet)).$$

We also denote by  $HH_\bullet(E_\bullet)$  resp.  $HC_\bullet(E_\bullet)$  the homology of  $CH_\bullet(E_\bullet)$  resp.  $CC_\bullet(E_\bullet)$ . Alternatively,  $CH_\bullet(E_\bullet)$  is the cone of the natural map

$$\text{id} - \sigma^\dagger : CH'_\bullet(j_l^* E_\bullet) \rightarrow CH_\bullet(j_l^* E_\bullet),$$

and in terms of the identifications (3.9), this map is the natural map induced by the map  $\mathbb{K}_1 \rightarrow \mathbb{K}_0$ . In particular, we have  $HH_\bullet(E_\bullet) \cong HH_\bullet(j_l^* E_\bullet)$ , and for any quasiisomorphism  $E_\bullet \rightarrow E'_\bullet$ , the corresponding map

$$(3.14) \quad CH_\bullet(E_\bullet) \rightarrow CH_\bullet(E'_\bullet)$$

of mixed complexes is a quasiisomorphism in the sense of Definition 2.4. We note that  $\text{Exp}$  commutes with the totalization  $\text{tot}$ , so that we have  $CC_\bullet(E_\bullet) \cong cc_\bullet(\text{Exp}(\mathbb{K}_\bullet(E_\bullet)))$ . We also have natural augmentation maps  $\gamma$ ,  $\alpha$  of (3.11) resp. (3.12).

**3.2 Periodic complexes – definitions.** We now assume given a complex  $E_\bullet$  in the category  $\text{Fun}(\Lambda_l, R)$ , take the mixed complex  $CH_\bullet(E_\bullet)$ , and consider the periodic expansions of Definition 2.3.

**Definition 3.2.** Assume given a complex  $E_\bullet$  in the category  $\text{Fun}(\Lambda_l, R)$ . Then the *periodic cyclic complex*  $CP_\bullet(E_\bullet)$  and the *co-periodic cyclic complex*  $\overline{CP}_\bullet(E_\bullet)$  are given by

$$(3.15) \quad CP_\bullet(E_\bullet) = \text{Per}(CH_\bullet(E_\bullet)), \quad \overline{CP}_\bullet(E_\bullet) = \overline{\text{Per}}(CH_\bullet(E_\bullet)).$$

The homology of the complexes  $CP_\bullet(E_\bullet)$  resp.  $\overline{CP}_\bullet(E_\bullet)$  are denoted by  $HP_\bullet(E_\bullet)$  resp.  $\overline{HP}_\bullet(E_\bullet)$ .

The complexes  $CP_\bullet(E_\bullet)$  and  $\overline{CP}_\bullet(E_\bullet)$  and their homology are the main objects of study in this paper. Unfortunately, there is no map from one complex to the other one. To be able to compare the two, we have to consider an additional functorial complex.

**Definition 3.3.** For any complex  $E_\bullet$  in  $\text{Fun}(\Lambda_l, R)$ , the *polynomial periodic cyclic complex*  $cp_\bullet(E_\bullet)$  is given by

$$cp_\bullet(E_\bullet) = \text{per}(CH_\bullet(E_\bullet)) \cong cc_\bullet(\text{per}(\mathbb{K}_\bullet(E_\bullet))).$$

Then for any complex  $E_\bullet$  in  $\text{Fun}(\Lambda_l, R)$ , the natural maps (2.2) induce natural maps

$$(3.16) \quad CP_\bullet(E_\bullet) \longleftarrow cp_\bullet(E_\bullet) \longrightarrow \overline{CP}_\bullet(E_\bullet)$$

from the polynomial periodic cyclic complex  $cp_\bullet(E_\bullet)$  to the periodic and co-periodic cyclic complexes. In general, neither of these maps is a quasi-isomorphism. Moreover, for any quasiisomorphism  $f : E_\bullet \rightarrow E'_\bullet$ , we can consider the corresponding maps

$$(3.17) \quad CP_\bullet(E_\bullet) \rightarrow CP_\bullet(E'_\bullet), \quad \overline{CP}_\bullet(E_\bullet) \rightarrow \overline{CP}_\bullet(E'_\bullet), \quad cp_\bullet(E_\bullet) \rightarrow cp_\bullet(E'_\bullet).$$

induced by the map (3.14). Then the first of these maps is a quasiisomorphism by Lemma 2.6, so that  $CP_\bullet(-)$  descends to a functor on the derived category  $\mathcal{D}(\Lambda_l, R)$ . However, this need not be true for the other two maps, so that neither  $cp_\bullet(-)$  nor  $\overline{CP}_\bullet(-)$  are defined on the level of the derived category, and one cannot study them by the standard techniques of homological algebra.

One way out of this difficulty is to notice that for any abelian category  $\mathcal{C}$ , complexes in  $\mathcal{C}$  strongly acyclic in the sense of Definition 2.8 form a

triangulated subcategory in the homotopy category of chain complexes. One can then invert strong quasiisomorphisms in  $C_*(\mathcal{C})$  and obtain the so-called *absolute derived category*  $\mathcal{D}_{abs}(\mathcal{C})$ . This is a triangulated category introduced and studied extensively by L. Positselski [P]. If a map  $E_* \rightarrow E'_*$  is a strong quasiisomorphism, then all the maps (3.17) are quasiisomorphisms, so that  $cp_*(-)$  and  $\overline{CP}_*(-)$  do descend to the co-derived category  $\mathcal{D}_{abs}(\Lambda_l, R)$  of the abelian category  $\text{Fun}(\Lambda_l, R)$ . However, absolute derived categories are at present not well enough understood for our purposes. So, for most of our results, we adopt a more conventional approach – we equip everything in sight with a filtration, and use filtered derived categories described in Subsection 1.2.

We recall that the cyclic complex functor  $cc_*(-)$  can be promoted to a functor  $cc^f$  of (1.12) from the category of filtered complexes in  $\text{Fun}(\Lambda_l, R)$  with a termwise-split filtration to the category of filtered complexes of  $R$ -modules. We will need to modify this slightly.

**Definition 3.4.** For any filtered complex  $\langle E_*, F^\bullet \rangle$  in  $\text{Fun}(\Lambda_l, R)$  with termwise-split filtration, the *standard filtration*  $F^\bullet$  on the complex  $cc_*(E_*)$  is given by

$$(3.18) \quad F^n cc_*(E_*) = F^{n-1} cc^f(E_*),$$

where the filtered complex  $cc^f(E_*)$  is as in (1.11).

In other words, we shift the filtration on  $cc^f(E_*)$  by 1 in the sense of (1.4). From now on, for any filtered complex  $E_*$  in  $\text{Fun}(\Lambda_l, R)$  with a termwise-split filtration, we will write simply  $cc_*(E_*)$  to mean  $cc_*(E_*)$  equipped with the standard filtration (3.18).

For any filtered complex  $E_*$  with termwise-split filtration, the filtration on  $E_*$  induces a termwise-split filtration on  $\mathbb{K}_*(E_*) = \mathbb{K}_* \otimes E_*$ , and therefore the standard filtration (3.18) induces a filtration on  $CH_*(E_*) = cc_*(\mathbb{K}_*(E_*))$ . This filtration is preserved by the differential  $B$  in the mixed complex  $CH_*(E_*)$ , thus induces filtrations (2.4) on  $CP_*(E)$ ,  $\overline{CP}_*(E)$  and  $cp_*(E)$ . We will also call them the *standard filtrations*.

Spelling out Definition 3.4, we see that for any complex  $E_*$  in  $\text{Fun}(\Lambda_l, R)$  equipped with a termwise-split filtration  $F^\bullet$ , the associated graded quotient of the mixed complex  $CH_*(E_*)$  with respect to the standard filtration is given by

$$(3.19) \quad \text{gr}_F^i CH_*(E_*) \cong \bigoplus_{m \geq 1} \mathbb{K}_*(\text{gr}_F^{i-m} E_*([m]))_{\sigma^\dagger}[m-1]$$

for any integer  $i$ , where  $\mathbb{K}_\bullet(-)$  is the mixed complex (2.9), and  $[m-1]$  in the right-hand side stands for the homological shift. The associated graded quotients of the periodic complexes are then given by (2.5). In particular, since the polynomial expansion functor commutes with arbitrary direct sums, we have

$$(3.20) \quad \mathrm{gr}_F^i cp_\bullet(E_\bullet) = \bigoplus_{m \geq 1} \tilde{C}_\bullet^\dagger(\mathbb{Z}/lm\mathbb{Z}, \mathrm{gr}_F^{i-m} E_\bullet([m]))[m-1],$$

where  $\tilde{C}_\bullet^\dagger(-)$  is the twisted Tate homology complex of (2.14) with respect to the cyclic group  $\mathbb{Z}/ml\mathbb{Z} = \mathrm{Aut}([m])$ . Moreover, if  $E_\bullet$  is concentrated in a finite range of filtered degrees, then the direct sum in (3.19) is in fact finite, so that we also have

$$(3.21) \quad \begin{aligned} \mathrm{gr}_F^i CP_\bullet(E_\bullet) &= \bigoplus_{m \geq 1} \check{C}_\bullet^\dagger(\mathbb{Z}/lm\mathbb{Z}, \mathrm{gr}_F^{i-m} E_\bullet([m]))[m-1], \\ \mathrm{gr}_F^i \overline{CP}_\bullet(E_\bullet) &= \bigoplus_{m \geq 1} \check{\check{C}}_\bullet^\dagger(\mathbb{Z}/lm\mathbb{Z}, \mathrm{gr}_F^{i-m} E_\bullet([m]))[m-1], \end{aligned}$$

where  $\check{C}_\bullet^\dagger(-)$  and  $\check{\check{C}}_\bullet^\dagger(-)$  are again the complexes of (2.14).

**3.3 Periodic complexes – first properties.** It turns out that standard filtrations are already useful when we take a complex  $E_\bullet$  in  $\mathrm{Fun}(\Lambda_l, R)$  and treat it as a filtered complex concentrated in filtered degree 0. Then by definition, the standard filtrations on  $cp_\bullet(E_\bullet)$ ,  $CP_\bullet(E_\bullet)$ ,  $\overline{CP}_\bullet(E_\bullet)$  are bounded below (actually, already the term  $F^0$  vanishes). However, while the standard filtration on  $cp_\bullet(E_\bullet)$  is exhaustive, the same need not be true for  $CP_\bullet(E_\bullet)$  or  $\overline{CP}_\bullet(E_\bullet)$ . In fact, let us introduce the following.

**Definition 3.5.** For any complex  $E_\bullet$  in  $\mathrm{Fun}(\Lambda_l, R)$ , the *restricted* periodic resp. co-periodic complexes  $CP_\bullet^f(E_\bullet)$ ,  $\overline{CP}_\bullet^f(E_\bullet)$  are given by

$$CP_\bullet^f(E_\bullet) = cc_\bullet(\mathrm{Per}(\mathbb{K}_\bullet(E_\bullet))), \quad \overline{CP}_\bullet^f(E_\bullet) = cc_\bullet(\overline{\mathrm{Per}}(\mathbb{K}_\bullet(E_\bullet))).$$

Then  $CP_\bullet^f(E_\bullet)$ ,  $\overline{CP}_\bullet^f(E_\bullet)$  are precisely the completions of  $CP_\bullet(E_\bullet)$  resp.  $\overline{CP}_\bullet(E_\bullet)$  with respect to the standard filtrations in the sense of Definition 1.1. Alternatively, up to a shift of filtration, they coincide with the extensions (1.11) of the functors  $CP_\bullet(-)$ ,  $\overline{CP}_\bullet(-)$  from  $\mathrm{Fun}(\Lambda_l, R)$  to complexes of  $R$ -modules, so that the notation is consistent. Since the filtration

on  $E_\bullet$  is bounded below, the standard filtrations are also bounded below, and we have natural functorial maps

$$(3.22) \quad CP_\bullet^f(E_\bullet) \rightarrow CP_\bullet(E_\bullet), \quad \overline{CP}_\bullet^f(E_\bullet) \rightarrow \overline{CP}_\bullet(E_\bullet).$$

Together with the natural maps

$$CP_\bullet^f(E_\bullet) \longleftarrow cp_\bullet(E_\bullet) \longrightarrow \overline{CP}_\bullet^f(E_\bullet)$$

obtained by applying  $cc_\bullet$  to the maps (2.2), the maps (3.22) fit into the following diagram

$$(3.23) \quad \begin{array}{ccccc} cp_\bullet(E_\bullet) & \xrightarrow{r} & \overline{CP}_\bullet^f(E_\bullet) & \xrightarrow{R} & \overline{CP}_\bullet(E_\bullet) \\ \parallel & & & & \\ cp_\bullet(E_\bullet) & \xrightarrow{l} & CP_\bullet^f(E_\bullet) & \xrightarrow{L} & CP_\bullet(E_\bullet), \end{array}$$

a refinement of (3.16). All the maps in this diagram are injective maps of complexes, and in general, none of them are quasiisomorphisms.

By (3.21) and Lemma 2.6, for any quasiisomorphism  $E_\bullet \rightarrow E'_\bullet$  of complexes in  $\text{Fun}(\Lambda_l, R)$ , the first map in (3.17) is a filtered quasiisomorphism with respect to the standard filtration, so that the corresponding map

$$CP_\bullet^f(E_\bullet) \rightarrow CP_\bullet^f(E'_\bullet)$$

is a quasiisomorphism. Therefore just as  $CP_\bullet(-)$ , the complex  $CP_\bullet^f(-)$  is defined on the level of the derived category  $\mathcal{D}(\Lambda_l, R)$ , and so is the map  $L$  of (3.23). Note that the map need not be an isomorphism even in the simplest examples such as the following.

**Lemma 3.6.** *Let  $E_\bullet$  be the constant functor  $R$  placed in homological degree 0. Then  $CP_\bullet(R)$  is quasiisomorphic to the free rank-1 module over the Laurent power series ring  $R((u))$ , while  $CP_\bullet^f(R)$  is quasiisomorphic to  $(R \overset{\text{L}}{\otimes}_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z}))((u))[1]$ . The map  $L$  of (3.23) fits into a distinguished triangle*

$$CP_\bullet^f(R) \xrightarrow{L} CP_\bullet(R) \longrightarrow CP_\bullet(R \otimes_{\mathbb{Z}} \mathbb{Q}) \longrightarrow$$

*Proof.* Let  $E_i = R$ ,  $i \geq 0$ , let  $e_i : E_i \rightarrow E_{i+1}$  be given by

$$e_i = \begin{cases} (i+1) \text{id}, & i = 2n, \\ 2 \text{id}, & i = 2n+1, \end{cases}$$

and consider the bicomplex  $E_{\bullet,\bullet}$  of Lemma 1.6. Then one immediately checks that the periodic bicomplex  $\text{per}_{\bullet,\bullet}(CH_{\bullet}(R))$  of (2.6) coincides on the nose with  $E_{\bullet,\bullet}((u))$ . To finish the proof, apply Lemma 1.6.  $\square$

As for the other functorial complexes of (3.23), then in the simple situation such as that of Lemma 3.6, they contain no new information. Indeed, one checks easily that for any complex  $E_{\bullet}$  concentrated in a finite range of homological degrees, the maps  $r$ ,  $l$ , and  $R$  are isomorphisms, so that the only non-trivial invariants of  $E_{\bullet}$  are  $CP_{\bullet}(E_{\bullet})$  and  $CP_{\bullet}^f(E_{\bullet}) \cong cp_{\bullet}(E_{\bullet}) \cong \overline{CP}^f(E_{\bullet}) \cong \overline{CP}_{\bullet}(E_{\bullet})$ .

For unbounded complexes, the situation is more difficult, and neither of the maps is even a quasiisomorphism without additional assumptions. The only obvious general fact is the following result.

**Lemma 3.7.** *For any complex  $E_{\bullet}$  in  $\text{Fun}(\Lambda_l, R)$ , we have quasiisomorphisms  $CP_{\bullet}^f(E_{\bullet}) \otimes \mathbb{Q} \cong cp_{\bullet}(E_{\bullet}) \otimes \mathbb{Q} \cong \overline{CP}_{\bullet}^f(E_{\bullet}) \otimes \mathbb{Q} \cong 0$ .*

*Proof.* Since all the complexes are complete with respect to the standard filtration, it suffices to check that all the associated graded quotients (3.20), (3.21) are acyclic. But once we tensor with  $\mathbb{Q}$ , the differential in the Tate homology complex becomes chain-homotopic to 0.  $\square$

To analyse  $\overline{CP}_{\bullet}(E_{\bullet})$ , we can also use the standard filtrations, but in a different way: take a complex  $E_{\bullet}$  in  $\text{Fun}(\Lambda_l, R)$ , and equip it with the stupid filtration instead of the trivial one (it is automatically termwise-split). Then we have the following result.

**Lemma 3.8.** *Assume that a complex  $E_{\bullet}$  in  $\text{Fun}(\Lambda_l, R)$  is equipped with a stupid filtration. Then the map*

$$R \circ r : cp_{\bullet}(E_{\bullet}) \rightarrow \overline{CP}_{\bullet}(E_{\bullet})$$

*is a filtered quasiisomorphism with respect to the standard filtrations, and  $\overline{CP}_{\bullet}(E_{\bullet})$  is complete.*

*Proof.* In the case of the stupid filtration, (3.19) provides an isomorphism

$$(3.24) \quad \text{gr}_F^i CH_{\bullet}(E_{\bullet}) \cong \bigoplus_{m \geq 1} \mathbb{K}_{\bullet}(E^{i-m}([m]))_{\sigma^+}[i-1]$$

for any integer  $i$ . In particular, for any  $i$ ,  $\text{gr}_F^i CH_{\bullet}(E_{\bullet})$  is concentrated in a finite range of cohomological degrees (degrees  $-i$  and  $1-i$ , to be exact).

Then  $R \circ r$  is a filtered quasiisomorphism by (2.5). Moreover, the completion of the complex  $cp_{\bullet}(E_{\bullet})$  can be computed as in Example 1.2, and it clearly coincides with  $\overline{CP}_{\bullet}(E_{\bullet})$ .  $\square$

**Corollary 3.9.** *For any complex  $E_{\bullet}$  in  $\text{Fun}(\Lambda, R)$ , the complex  $\overline{CP}_{\bullet}(E_{\bullet}) \otimes \mathbb{Q}$  is acyclic.*

*Proof.* Equip  $E_{\bullet}$  with the stupid filtration as in Lemma 3.8, so that the tensor product  $\overline{CP}_{\bullet}(E_{\bullet}) \otimes \mathbb{Q} \cong \overline{CP}_{\bullet}(E_{\bullet} \otimes \mathbb{Q})$  is complete with respect to the standard filtration. Then it suffices to check that its associated graded quotient is acyclic. By (3.24), this associated graded quotient is a sum of Tate cohomology complexes, and as in Lemma 3.7, these become acyclic after tensoring with  $\mathbb{Q}$ .  $\square$

**3.4 Representing objects.** Assume given a small category  $I$  and a ring  $R$ , and denote by  $I^o$  the opposite category. Then for any objects  $E \in \text{Fun}(I, R)$ ,  $M \in \text{Fun}(I^o, \mathbb{Z})$ , the *tensor product*  $E \otimes_I M$  is the cokernel of the natural map

$$\bigoplus_{f:i \rightarrow i'} E(i) \otimes M(i') \xrightarrow{E(f) \otimes \text{id} - \text{id} \otimes M(i')} \bigoplus_{i \in I} E(i) \otimes M(i),$$

where the sum in the right-hand side is over all objects  $i \in I$ , and the sum in the left-hand side is over all morphisms  $f : i \rightarrow i'$  in  $I$ . This tensor product and its derived functor  $- \overset{\text{L}}{\otimes}_I -$  are very convenient for representing functors from  $\text{Fun}(I, R)$  to  $R$ -modules or complexes of  $R$ -modules. For example, for any  $E \in \text{Fun}(I, R)$ , we have a natural identification

$$(3.25) \quad E \overset{\text{L}}{\otimes}_I \mathbb{Z} \cong H_{\bullet}(I, E),$$

where  $\mathbb{Z} \in \text{Fun}(I^o, \mathbb{Z})$  is the constant functor. On the other hand, for any object  $i \in I$ , we have a natural Yoneda-type identification

$$(3.26) \quad E(i) \cong E \otimes_I \mathbb{Z}_i \cong E \overset{\text{L}}{\otimes}_I \mathbb{Z}_i,$$

where  $\mathbb{Z}_i \in \text{Fun}(I^o, \mathbb{Z})$  is the representable functor given by

$$\mathbb{Z}_i(i') = \mathbb{Z}[I(i', i)],$$

with  $I(i', i)$  standing for the set of morphisms  $f : i' \rightarrow i$  in  $I$ . Thus to compute homology  $H_{\bullet}(I, E)$  by an explicit complex functorial in  $E$ , it suffices to

find a resolution of the constant functor  $\mathbb{Z} \in \text{Fun}(I^o, \mathbb{Z})$  by sums of objects  $\mathbb{Z}_i$ ,  $i \in I$ .

Applying this formalism to cyclic homology, we can represent the functorial complexes  $cc_\bullet(-)$ ,  $CH_\bullet(-)$ ,  $CC_\bullet(-)$  of Subsection 3.1. Namely, fix an integer  $l \geq 1$ , for any  $i \geq 0$ , let

$$Q_i = \mathbb{Z}_{[i+1]} \in \text{Fun}(\Lambda_l^o, \mathbb{Z}),$$

and let  $q_i = (Q_i)_{\sigma^\dagger}$  be cokernel of the natural map  $\text{id} - \sigma^\dagger : Q_i \rightarrow Q_i$ , where  $\sigma$  is the generator of the cyclic group  $\mathbb{Z}/(i+1)l\mathbb{Z} = \text{Aut}([i+1])$ , and  $\sigma^\dagger$  is as in Subsection 2.2. Then (3.3) defines a differential  $d : Q_{\bullet+1} \rightarrow Q_\bullet$  turning  $Q_\bullet$  into a complex, and it descends to a differential  $d : q_{\bullet+1} \rightarrow q_\bullet$ . By (3.26), we then have

$$cc_\bullet(E) \cong (E \otimes_{\Lambda_l} Q_\bullet)_\sigma \cong E \otimes_{\Lambda_l} q_\bullet$$

for any object  $E \in \text{Fun}(\Lambda_l, R)$ . Analogously, (3.4) defines a differential  $d' : Q_{\bullet+1} \rightarrow Q_\bullet$ . Denote the resulting complex by  $Q'_\bullet$ , and denote by  $P_\bullet$  the cone of the natural map of complexes

$$(3.27) \quad Q'_\bullet \rightarrow Q_\bullet$$

given by  $(\text{id} - \sigma^\dagger)$  termwise. Then  $P_\bullet$  is a mixed complex in  $\text{Fun}(\Lambda_l^o, \mathbb{Z})$ , and we have

$$CH_\bullet(E) \cong E \otimes_{\Lambda_l} P_\bullet$$

for any  $E \in \text{Fun}(\Lambda_l, R)$ .

**Lemma 3.10.** *The mixed complex  $P_\bullet$  is a mixed resolution of the constant functor  $\mathbb{Z} \in \text{Fun}(\Lambda_l^o, \mathbb{Z})$  in the sense of Definition 2.12.*

*Proof.* The statement is essentially [FT, Appendix, A3], but we reproduce the proof for the convenience of the reader. By (3.26), it suffices to prove that for any  $[n] \in \Lambda_l$ , the mixed complex  $CH_\bullet(\mathbb{Z}_{[n]})$  is a mixed resolution of the abelian group  $\mathbb{Z}$ . Fix  $[n]$ , and denote by  $E_\bullet$ ,  $E'_\bullet$  the cokernel resp. kernel of the natural map

$$CH'_\bullet(j_l^* \mathbb{Z}_{[n]}) \xrightarrow{\text{id} - \sigma^\dagger} CH_\bullet(j_l^* \mathbb{Z}_{[n]})$$

whose cone is the complex  $CH_\bullet(\mathbb{Z}_{[n]})$ . Then for any  $[m] \in \Lambda_l$ , the action of the cyclic group  $\mathbb{Z}/ml\mathbb{Z} = \text{Aut}([m])$  on the set of morphisms  $\Lambda_l([n], [m])$  is free. Therefore for any  $i \geq 0$ , the differential  $B$  in the mixed complex  $CH_\bullet(\mathbb{Z}_{[n]})$  is an isomorphism between the cokernel  $E_i$  and the kernel  $E'_i$



of the endomorphism  $\text{id} - \sigma^\dagger$  of the  $\mathbb{Z}[\mathbb{Z}/l(i+1)\mathbb{Z}]$ -module  $\mathbb{Z}_{[n]}([i+1]) = \mathbb{Z}(\Lambda_l([n], [i+1]))$ . We conclude that  $B$  factors through an isomorphism  $E_\bullet \rightarrow E'_\bullet$ , so that it suffices to prove that  $E_\bullet$  is a resolution of  $\mathbb{Z}$ . But  $E_\bullet$  is precisely the standard chain complex of the elementary simplex  $\Delta_{[n]}^o \in \Delta^o \text{Sets}$ .  $\square$

As a corollary of Lemma 3.10, we see that the expansion  $\text{Exp}(P_\bullet)$  of the complex  $P_\bullet$  is a resolution of the constant functor  $\mathbb{Z} \in \text{Fun}(\Lambda_l, \mathbb{Z})$ , so that by (3.25), for any  $E \in \text{Fun}(\Lambda_l, R)$ , the complex

$$(3.28) \quad CC_\bullet(E) = \text{Exp}(CH_\bullet(E)) \cong \text{Exp}(E \otimes_{\Lambda_l} P_\bullet) \cong E \otimes_{\Lambda_l} \text{Exp}(P_\bullet)$$

computes the homology  $H_\bullet(\Lambda_l, R)$  of the category  $\Lambda_l$  with coefficients in  $E$ , and more generally, we have  $HC_\bullet(E_\bullet) \cong H_\bullet(\Lambda_l, E_\bullet)$  for any complex  $E_\bullet$  in  $\text{Fun}(\Lambda_l, R)$  (this is of course well-known, see e.g. [FT, Appendix, Corollary A3.2]). Lemma 3.6 then implies another well-known fact, namely, that the cohomology  $H^*(\Lambda_l, \mathbb{Z})$  is the free algebra  $\mathbb{Z}[u]$  in one generator  $u$  of degree 2, and the cohomology class of the mixed resolution  $P_\bullet$  is exactly  $u$ .

For some applications of the tensor product formalism, one does not even need to know the exact shape of the representing complexes. For example, we will need the following easy observation.

**Lemma 3.11.** *For any complex  $E_\bullet$  in  $\text{Fun}(\Lambda_l, R)$ , the map*

$$(3.29) \quad \alpha : CC_\bullet(\mathbb{K}_\bullet(E_\bullet)) \rightarrow cc_\bullet(\mathbb{K}_\bullet(E_\bullet)) \cong CH_\bullet(E_\bullet)$$

*induced by the augmentation map (3.12) is a quasiisomorphism.*

*Proof.* The functor  $CH_\bullet(-)$  is represented by the complex  $P_\bullet$  of Lemma 3.10, and the functor  $CC_\bullet(\mathbb{K}_\bullet(E_\bullet))$  is also obviously represented by a complex  $M_\bullet$  of projective objects in  $\text{Fun}(\Lambda_l^o, \mathbb{Z})$ . Both  $P_\bullet$  and  $M_\bullet$  are bounded from above (by 0, but this is not important). The map (3.29) is induced by a map of complexes

$$a : M_\bullet \rightarrow P_\bullet.$$

Since for any  $[n] \in \Lambda_l$ , both  $\mathbb{K}_0([n])$  and  $\mathbb{K}_1([n])$  are free  $\mathbb{Z}[\mathbb{Z}/nl\mathbb{Z}]$ -modules,  $H_i(\mathbb{Z}/nl\mathbb{Z}, \mathbb{K}_0(E)) = H_i(\mathbb{Z}/nl\mathbb{Z}, \mathbb{K}_1(E)) = 0$  for any  $i \geq 1$  and any object  $E$  in  $\text{Fun}(\Lambda_l, R)$ . Therefore the map (3.29) is a quasiisomorphism when  $E_\bullet$  is concentrated in a single homological degree. By (3.26), this means that  $a$  is a quasiisomorphism after evaluating at any object  $[n] \in \Lambda_l^o$ , thus a quasiisomorphism of complexes in  $\text{Fun}(\Lambda_l^o, \mathbb{Z})$ . But since both  $M_\bullet$  and  $P_\bullet$  are complexes of projective objects bounded from above,  $a$  must then be a chain-homotopy equivalence. Therefore the map (3.29) is also a chain-homotopy equivalence, thus a quasiisomorphism for any  $E_\bullet$ .  $\square$

Observe now that in fact, both the differentials (3.3) and (3.4) only involve the standard face maps – that is, injective maps in  $\Delta$ . These correspond to surjective maps in  $\Delta^\circ$  and in  $\Lambda_l$ . Therefore if we denote by  $\overline{\Lambda}_l \subset \Lambda_l$  the subcategory of surjective maps, and let  $e : \overline{\Lambda}_l \rightarrow \Lambda_l$  be the natural embedding, then for any  $E \in \text{Fun}(\Lambda_l, R)$ , the mixed complex  $CH_\bullet(E)$  only depends on the restriction  $e^*E \in \text{Fun}(\overline{\Lambda}_l, R)$ . In terms of representing objects, denote

$$\overline{Q}_i = \mathbb{Z}_{[i+1]} \in \text{Fun}(\overline{\Lambda}_l^\circ, \mathbb{Z})$$

for any  $i \geq 0$ . Then we have  $Q_i \cong e_! \overline{Q}_i$ , the differentials  $d, d' : Q_{\bullet+1} \rightarrow Q_\bullet$  are induced by differentials  $d, d' : \overline{Q}_{\bullet+1} \rightarrow \overline{Q}_\bullet$ , and we have

$$Q_\bullet \cong e_! \overline{Q}_\bullet, \quad Q'_\bullet \cong e_! \overline{Q}'_\bullet$$

for some canonical complexes  $\overline{Q}_\bullet, \overline{Q}'_\bullet$  in  $\text{Fun}(\overline{\Lambda}_l^\circ, \mathbb{Z})$ . Moreover, since  $\sigma \in \text{Aut}([m])$  is surjective for any  $[m] \in \Lambda_l$ , the map (3.27) is induced by a map  $\text{id} - \sigma^\dagger : \overline{Q}'_\bullet \rightarrow \overline{Q}_\bullet$ , and we have

$$P_\bullet \cong e_! \overline{P}_\bullet,$$

where  $\overline{P}_\bullet$  is the cone of  $\text{id} - \sigma$ .

**Lemma 3.12.** *The complex  $\overline{P}_\bullet$  is a mixed resolution of the constant functor  $\mathbb{Z} \in \text{Fun}(\overline{\Lambda}_l^\circ, \mathbb{Z})$  in the sense of Definition 2.12.*

*Proof.* Literally the same as Lemma 3.10, except that the complex  $E_\bullet$  is not the standard chain complex of the elementary simplex  $\Delta_{[n]}^\circ \in \Delta^\circ \text{Sets}$  but rather, its normalized chain complex.  $\square$

## 4 Projections and subdivisions.

**4.1 Edgewise subdivision.** Fix an integer  $l \geq 1$ , and consider the functor  $i_l$  of (3.1). Recall the following result.

**Lemma 4.1.** *For any ring  $R$  and any  $E \in \mathcal{D}(\Lambda, R)$ , the natural map*

$$H_\bullet(\Lambda_l, i_l^* E) \rightarrow H_\bullet(\Lambda, E)$$

*induced by the functor  $i_l$  is an isomorphism.*  $\square$

This result is known as “edgewise subdivision” and goes back to Segal and Quillen (a short proof with exactly the same notation as in this paper can be found in [Ka2, Lemma 1.14]).

The homology groups  $H_*(\Lambda, -)$ ,  $H_*(\Lambda_l, -)$  can be computed by the cyclic complexes  $CC_*(-)$ , and it is natural to ask whether one can lift the edgewise subdivision isomorphism to a map of complexes. To do this, one can use the representing complexes  $P_*$  of Subsection 3.4. By Lemma 3.10, both  $P_*$  and  $i_l^* P_*$  are mixed resolutions of the constant functor  $\mathbb{Z} \in \text{Fun}(\Lambda_l^o, \mathbb{Z})$ . Moreover, Lemma 4.1 shows that the natural map  $i_l^* : H^2(\Lambda^o, \mathbb{Z}) \rightarrow H^2(\Lambda_l^o, \mathbb{Z})$  is an isomorphism, so that both resolutions have the same cohomology class  $u = i_l^* u$ . Since  $P_i$  is a projective object for any  $i$ , we can apply Lemma 2.13 and obtain a quasiisomorphism

$$\nu_l : \text{Exp}(P_*) \rightarrow i_l^* \text{Exp}(P_*).$$

Applying (3.28), for any complex  $E_*$  in  $\text{Fun}(\Lambda, R)$ , we obtain a functorial map

$$(4.1) \quad \nu_l : CC_*(i_l^* E_*) \rightarrow CC_*(E_*)$$

realizing the edgewise subdivision isomorphism on the chain level.

Moreover, we can do this construction using the complexes  $\overline{P}_*$  instead of the complexes  $P_*$  and Lemma 3.12 instead of Lemma 3.10. Indeed, the functor  $i_l$  induces a functor  $\bar{i}_l : \overline{\Lambda}_l \rightarrow \overline{\Lambda}$  such that  $e \circ \bar{i}_l \cong i_l \circ e$ , and since the adjunction map

$$\overline{P}_* \rightarrow e^* e_l \overline{P}_* \cong e^* P_*$$

is a quasiisomorphism,  $\overline{P}_*$  and  $\bar{i}_l^* \overline{P}_*$  are also mixed resolutions of the constant functor  $\mathbb{Z}$  with the same cohomology class. Therefore Lemma 2.13 provides a natural map

$$(4.2) \quad \nu_l : \text{Exp}(\overline{P}_*) \rightarrow i_l^* \text{Exp}(\overline{P}_*).$$

This map induces a map

$$\nu_l : CC_*(\bar{i}_l^* \overline{E}_*) \rightarrow CC_*(\overline{E}_*)$$

for any complex  $\overline{E}_*$  in  $\text{Fun}(\overline{\Lambda}, R)$ . In general, this map is *not* a quasiisomorphism; however, if  $\overline{E}_* = e^* E_*$  for some complex  $E_*$  in  $\text{Fun}(\Lambda, R)$ , it is a quasiisomorphism since it coincides with the map (4.1).

Using  $\overline{P}_*$  instead of  $P_*$  has the following advantage. By (2.3), a map  $\nu$  provided by Lemma 2.13 uniquely extends to a  $u$ -equivariant map  $\nu : \text{Per}(P_*) \rightarrow \text{Per}(M_*)$ . In particular, the map (4.1) extends to a map

$$(4.3) \quad \nu_l : CP_*(i_l^* E_*) \rightarrow CP_*(E_*)$$

for any complex  $E_\bullet$  in  $\text{Fun}(\Lambda, R)$ .

**Lemma 4.2.** *Assume given a complex  $E_\bullet$  in  $\text{Fun}(\Lambda, R)$  equipped with a termwise-split filtration, and assume that the map (4.1) is induced by a map (4.2). Then the corresponding map  $\nu_l$  of (4.3) sends  $cp_\bullet(i_l^* E_\bullet) \subset CP_\bullet(i_l^* E_\bullet)$  into  $cp_\bullet(E_\bullet) \subset CP_\bullet(E_\bullet)$ , and induces a filtered map*

$$(4.4) \quad \nu_l : cp_\bullet(i_l^* E_\bullet^{[l]}) \rightarrow cp_\bullet(E_\bullet)^{[l]},$$

where the complexes  $cp_\bullet(-)$  are equipped with standard filtrations of Subsection 3.2, and  $[l]$  stand for the filtrations rescaled by  $l$ , as in (1.4).

*Proof.* It obviously suffices to prove both claims when  $E_\bullet$  is a single object  $E$  concentrated in a single filtered degree, say degree  $n$ . Let  $m$  be the integer such that  $lm \leq n < l(m+1)$ , so that  $E^{[l]}$  is concentrated in the filtered degree  $m$ . Then the standard filtration on  $CP_\bullet(i_l^* E_\bullet)$  is the shift by  $m$  of the filtration induced by the filtrations on  $\overline{Q}_\bullet, \overline{Q}_\bullet$  assigning filtered degree  $-(i+1)$  to  $\overline{Q}_i$ . Analogously, the standard filtration on  $CP_\bullet$  is obtained by assigning filtered degree  $-(i+1)$  to  $\overline{Q}_i$  and shifting by  $n$ . Then to prove the second claim, it suffices to show that  $\text{Hom}(\overline{Q}_i, i_l^* \overline{Q}_j) = 0$  for  $j+1-n > l(i+1-m)$ . Since  $n \geq lm$ , it suffices to let  $n = m = 0$ . Then the claim is clear – by Yoneda Lemma, we have

$$\text{Hom}(\overline{Q}_i, i_l^* \overline{Q}_j) \cong i_l^* \overline{Q}_j([i+1]) \cong \overline{Q}_j([l(i+1)]) \cong \mathbb{Z}[\overline{\Lambda}([l(i+1)], [j+1])],$$

and the set  $\overline{\Lambda}([l(i+1)], [j+1])$  is empty unless  $j+1 \leq l(i+1)$ . The first claim immediately follows from the second, since  $cp_\bullet(E) \cong CP^f_\bullet(E)$  is then the filtered completion of  $CP_\bullet(E)$ , and analogously for  $cp_\bullet(i_l^* E)$ .  $\square$

Note that since all the rescalings of a given filtration are commensurable, Lemma 3.8 shows that if we equip a complex  $E_\bullet$  with the stupid filtration, the completion of the target of the map (4.4) coincides with  $\overline{CP}_\bullet(E_\bullet)$ . However, the induced filtration on  $i_l^* E_\bullet^{[l]}$  is not the stupid filtration because of the rescaling. To adjust for this, we introduce the following.

**Definition 4.3.** For any complex  $E_\bullet$  in  $\text{Fun}(\Lambda_l, R)$  and any integer  $n$ , the complex  $\overline{CP}^{[n]}_\bullet(E_\bullet)$  is the completion of the complex  $cp_\bullet(E_\bullet)$  with respect to the standard filtration of Definition 3.4 corresponding to the  $n$ -th rescaling  $F_{[n]}^\bullet$  of the stupid filtration on  $E_\bullet$ . The homology of the complex  $\overline{CP}^{[n]}_\bullet(E_\bullet)$  is denoted  $\overline{HP}^{[n]}_\bullet(E_\bullet)$ .

Then if  $n = 1$ ,  $\overline{CP}_\bullet^{[1]}(E_\bullet)$  coincides with  $\overline{CP}_\bullet(E_\bullet)$  by Lemma 3.8, but for  $n > 1$  this need not be the case. Nevertheless, the map (4.4) induces a map

$$(4.5) \quad \nu_l : \overline{CP}^{[l]}(i_l^* E_\bullet) \rightarrow \overline{CP}(E_\bullet)$$

for any complex  $E_\bullet$  in  $\text{Fun}(\Lambda, R)$ .

**4.2 Filtered refinement.** Assume now that  $l = p$  is an odd prime that annihilates our base ring  $R$ . Then we have the following filtered counterpart of Lemma 4.1.

**Proposition 4.4.** *Assume that  $l = p$  is an odd prime such that  $pR = 0$ . Then for any map  $\nu_p$  of (4.2) and any filtered complex  $E_\bullet$  in  $\text{Fun}(\Lambda, R)$ , the corresponding filtered map (4.4) provided by Lemma 4.2 is a filtered quasi-isomorphism.*

In order to prove this, we need a preliminary lemma. Let  $k = \mathbb{Z}/p\mathbb{Z}$  be the prime field corresponding to  $p$ , fix an integer  $n$ , and consider the group algebra  $k[\mathbb{Z}/np\mathbb{Z}]$ . We have a natural augmentation map

$$\text{aug} : k[\mathbb{Z}/np\mathbb{Z}] \rightarrow k$$

sending the generator  $\sigma \in \mathbb{Z}/np\mathbb{Z}$  to  $(-1)^{np}$ , so that  $\text{aug}(\sigma^\dagger) = 1$ . Moreover, assume given two elements  $a, b \in k[\mathbb{Z}/np\mathbb{Z}]$  such that

$$(4.6) \quad \begin{aligned} (1 - \sigma^\dagger)a &= b(1 - \sigma^\dagger), \\ (1 + \sigma^\dagger + \cdots + (\sigma^\dagger)^{np-1})b &= b(1 + \sigma^\dagger + \cdots + (\sigma^\dagger)^{np-1}). \end{aligned}$$

Then for any  $k[\mathbb{Z}/pn\mathbb{Z}]$ -module  $E$ , we have a functorial  $u$ -equivariant map of twisted Tate homology complexes

$$(4.7) \quad f(a, b) : \check{C}_\bullet^\dagger(\mathbb{Z}/np\mathbb{Z}, E) \rightarrow \check{C}_\bullet^\dagger(\mathbb{Z}/np\mathbb{Z}, E)$$

given by

$$f(a, b) = \begin{cases} a & \text{on } C_{2i}^\dagger(\mathbb{Z}/np, E), \\ b & \text{on } C_{2i+1}^\dagger(\mathbb{Z}/np, E) \end{cases}$$

for any integer  $i$ , and any  $u$ -equivariant functorial map (4.7) arises in this way.

**Lemma 4.5.** *In the assumptions above, the following are equivalent:*

- (i)  $\text{aug}(a) \neq 0$ ,

(ii)  $\text{aug}(b) \neq 0$ ,

(iii)  $f(a, b)$  is a quasiisomorphism for any  $E$ .

*Proof.* Note that (4.6) implies that  $(1 - \sigma^\dagger)(a - b) = 0$ , and since  $\mathbb{Z}/pn\mathbb{Z}$  has no homology with coefficients in its regular representation, this implies that  $(a - b) = (1 + \sigma^\dagger + \cdots + (\sigma^\dagger)^{np-1})h$  for some  $h \in k[\mathbb{Z}/np\mathbb{Z}]$ . Since  $\text{aug}(1 + \sigma^\dagger + \cdots + (\sigma^\dagger)^{np-1}) = np = 0$ , we have  $\text{aug}(a) = \text{aug}(b)$ , so that (i) and (ii) are equivalent. Since any group acts trivially on its homology,  $\mathbb{Z}/pn\mathbb{Z}$  acts on the homology of the twisted Tate complex  $\check{C}^\dagger(E)$  via the sign representation, so that the map  $f(a, b)$  acts on this homology by  $\text{aug}(a)$  in even degrees and  $\text{aug}(b)$  in odd degrees. Thus (i) and (ii) together imply (iii). Conversely, to deduce (i) and (ii) from (iii), take  $E$  to be the trivial representation  $k$ .  $\square$

*Proof of Proposition 4.4.* By (3.20), it suffices to prove the claim when  $E_\bullet$  is a single object  $E$  placed in a single filtered degree  $i$ . Since  $R$  is  $p$ -torsion, the twisted Tate homology complexes  $\check{C}_\bullet^\dagger(\mathbb{Z}/m\mathbb{Z}, M)$  are acyclic for any  $m$  not divisible by  $p$  and any  $R[\mathbb{Z}/m\mathbb{Z}]$ -module  $M$ , so that by (3.20),  $\text{gr}_F^m \text{cp}_\bullet(E)$  is acyclic unless  $m - i$  divides  $p$ . Therefore for any integer  $m$ , we have a quasiisomorphism

$$\text{gr}_F^m \text{cp}_\bullet(E_\bullet)^{[p]} \cong \text{gr}_F^{pn+i} \text{cp}_\bullet(E_\bullet),$$

where  $n$  is the integer such that  $pn \leq m - i < p(n + 1)$ . For any  $n \geq 1$ , the map  $\nu_p$  induces a functorial  $u$ -equivariant map

$$\text{gr}_F^{i-np} \nu_p : \check{C}_\bullet^\dagger(\mathbb{Z}/np\mathbb{Z}, i_p^* E([n])) \rightarrow \check{C}_\bullet^\dagger(\mathbb{Z}/np\mathbb{Z}, E([np]))[-n(p-1)],$$

and we have to prove that all these maps are quasiisomorphisms. Since  $i_p^* E([n]) \cong E([np])$ , and the Tate complex is 2-periodic, thus  $n(p-1)$ -periodic, we have

$$(4.8) \quad \text{gr}_F^{i-np} \nu_p = f(a_n, b_n)$$

for some  $a_n, b_n \in \mathbb{Z}[\mathbb{Z}/np\mathbb{Z}]$  satisfying (4.6). By Lemma 4.5, we then have to prove that  $\text{aug}(a_n)$  and  $\text{aug}(b_n)$  are invertible for any  $n \geq 1$ .

We note that by construction,  $a_n$  and  $b_n$  are universal constants that only depend on  $p$ , and do not depend on  $R$  and  $E$ . Thus we can take  $R = k$ , and take as  $E$  the constant functor  $k \in \text{Fun}(\Lambda, k)$  placed in filtered degree 0. Then  $i_p^* E$  is also constant, so that all the terms in the complexes  $CH_\bullet(E)$ ,

$CH'_\bullet(E)$ ,  $CH_\bullet(i_p^*E)$ ,  $CH'_\bullet(i_p^*E)$  are identified with  $k$ , and one immediately checks that the differentials  $d$ ,  $d'$  are given by

$$d = \begin{cases} \text{id} & \text{on } CH_{2i-1}(i_p^*E), \\ 0 & \text{on } CH_{2i}(i_p^*E), \end{cases} \quad d' = \begin{cases} 0 & \text{on } CH'_{2i-1}(i_p^*E), \\ \text{id} & \text{on } CH'_{2i}(i_p^*E), \end{cases}$$

as in Lemma 3.6. Therefore for any  $i \geq 1$ , we have

$$\begin{aligned} \text{aug}(b_{2i-1}) &= \text{aug}(b_{2i-1}) \circ d' = d \circ \text{aug}(a_{2i}), \\ \text{aug}(a_{2i}) &= \text{aug}(a_{2i}) \circ d = d' \circ \text{aug}(b_{2i+1}), \end{aligned}$$

and by Lemma 4.5 and induction on  $i$ , it suffices to prove that  $\text{aug}(a_1) \in k$  is non-zero. By Lemma 3.6, this amounts to checking that the map

$$\nu_p : cp_\bullet(i_p^*E) \cong \overline{CP}_\bullet(i_p^*E) \rightarrow cp_\bullet(E) \cong \overline{CP}_\bullet(E)$$

is a quasiisomorphism, and moreover, by the same Lemma 3.6, we have quasiisomorphisms  $\overline{CP}_\bullet(i_p^*E) \cong CP_\bullet(i_p^*E)$  and  $\overline{CP}_\bullet(E) \cong CP_\bullet(E)$ . Then we are done by Lemma 4.1.  $\square$

In particular, we can equip any complex  $E_\bullet$  in  $\text{Fun}(\Lambda, R)$  with the stupid filtration. Then the map  $\nu_p$  gives by completion the map (4.5), and Proposition 4.4 shows that it is a quasiisomorphism (provided  $l = p$  is an odd prime that annihilates  $R$ ). We also have the following statement for the restricted complexes  $CP^f(-)$ ,  $\overline{CP}^f(-)$  of Definition 3.5.

**Corollary 4.6.** *Under the assumptions of Proposition 4.4, the map  $\nu_p$  extends to quasiisomorphisms*

$$(4.9) \quad CP_\bullet^f(i_p^*E_\bullet) \cong CP_\bullet^f(E_\bullet), \quad \overline{CP}_\bullet^f(i_p^*E_\bullet) \cong \overline{CP}_\bullet^f(E_\bullet).$$

*Proof.* Place the complex  $E_\bullet$  in filtered degree 0, and equip the complexes  $cp_\bullet(E_\bullet)$ ,  $cp_\bullet(i_p^*E_\bullet)$  with the corresponding standard filtrations  $F^\bullet$ . Moreover, assume given an additional filtration  $W^\bullet$  on  $E_\bullet$ , and equip  $i_p^*E_\bullet$  with the induced filtration. Then since  $i_p^*$  is an exact functor, we have  $i_p^* \text{gr}_W^\bullet E_\bullet \cong \text{gr}_W^\bullet i_p^*E_\bullet$ , and for any  $n \geq 1$ , the quasiisomorphism

$$F^{-n}\nu_p : F^{-n}cp_\bullet(i_p^*E_\bullet) \cong F^{-n}cp_\bullet(E_\bullet)$$

of Proposition 4.4 is a filtered quasiisomorphism with respect to filtrations induced by  $W^\bullet$ . Therefore if we denote

$$cp_\bullet^W(E_\bullet) = \lim_{\substack{\longrightarrow \\ n}} \widehat{F^{-n}cp_\bullet(E_\bullet)}, \quad cp_\bullet^W(i_p^*E_\bullet) = \lim_{\substack{\longrightarrow \\ n}} \widehat{F^{-n}cp_\bullet(i_p^*E_\bullet)},$$

where the completions in the right-hand side are taken with respect to the filtration induced by  $W^\bullet$ , then  $\nu_p$  gives a quasiisomorphism

$$\nu_p : cp_\bullet^W(i_p^* E_\bullet) \cong cp_\bullet^W(E_\bullet).$$

It remains to notice that if  $W^\bullet$  is the canonical resp. stupid filtration on  $E_\bullet$ , then by Lemma 2.11 and (3.20), (3.21), the complex  $cp_\bullet^W(E_\bullet)$  coincides with  $CP^f(E_\bullet)$  resp.  $\overline{CP}^f(E_\bullet)$ .  $\square$

**4.3 Projections.** Consider now the second functor of (3.1), namely, the projection  $\pi_l : \Lambda_l \rightarrow \Lambda$ . By definition, for any  $E \in \mathcal{D}(\Lambda_l, R)$ , we have

$$H_\bullet(\Lambda_l, E) \cong H_\bullet(\Lambda, L^\bullet \pi_{p!} E).$$

However, unlike  $i_l^*$ , the functor  $\pi_l^*$  does not induce an isomorphism on homology and cohomology. In fact, the natural map

$$\pi_l^* : H^\bullet(\Lambda_l, R) \rightarrow H^\bullet(\Lambda, R)$$

sends the generator  $u$  of the algebra  $H^\bullet(\Lambda_l, R) \cong R[u]$  to  $lu$ , where  $u$  is the generator of  $R[u] \cong H^\bullet(\Lambda, R)$ .

In particular, assume that  $l = p$  is a prime such that  $pR = 0$ . Then  $\pi_p^* u$  vanishes. Therefore by adjunction, for any  $E \in \mathcal{D}(\Lambda, R)$  such that  $E \cong L^\bullet \pi_{p!} E'$  for some  $E' \in \mathcal{D}(\Lambda_p, R)$ , the connecting differential in the long exact sequence

$$(4.10) \quad H_\bullet(\Lambda, E)[1] \longrightarrow H_\bullet(\Delta^o, j^* E) \xrightarrow{\gamma} H_\bullet(\Lambda, R) \longrightarrow$$

vanishes, so that the natural map  $\gamma : H_\bullet(\Delta^o, j^* E) \rightarrow H_\bullet(\Lambda, E)$  admits a splitting

$$(4.11) \quad H_\bullet(\Lambda, E) \rightarrow H_\bullet(\Delta^o, j^* E).$$

The goal of this subsection is to refine this observation to a statement on the level of complexes, similar to Proposition 4.4.

For any complex  $E_\bullet$  in  $\text{Fun}(\Lambda_p, R)$ ,  $\pi_{p!} K_\bullet(E_\bullet)$  is a mixed complex in  $\text{Fun}(\Lambda, R)$ . Denote by

$$(4.12) \quad \pi_{pb} E_\bullet = \text{per}(\pi_{p!} \mathbb{K}_\bullet(E_\bullet))$$

its polynomial periodic expansion. By virtue of the identification (3.7), (3.3) gives a canonical isomorphism

$$(4.13) \quad cp_\bullet(E_\bullet) \cong cc_\bullet(\pi_{pb} E_\bullet).$$



Moreover, denote

$$cph_{\bullet}(E_{\bullet}) = CH_{\bullet}(\pi_{pb}E_{\bullet}) = cc_{\bullet}(\mathbb{K}_{\bullet}(\pi_{pb}E_{\bullet})).$$

Then by the projection formula, we have

$$(4.14) \quad \mathbb{K}_{\bullet}(\pi_{pb}E_{\bullet}) \cong \pi_{pb}E_{\bullet} \otimes \mathbb{K}_{\bullet} \cong \pi_{pb}(E_{\bullet} \otimes \pi_p^*\mathbb{K}_{\bullet}),$$

so that we have a natural isomorphism

$$(4.15) \quad cph_{\bullet}(E_{\bullet}) \cong cp_{\bullet}(E_{\bullet} \otimes \pi_p^*\mathbb{K}_{\bullet}),$$

and the natural map (3.11) gives a natural functorial map

$$(4.16) \quad \gamma : cph_{\bullet}(E_{\bullet}) \rightarrow cp_{\bullet}(E_{\bullet}),$$

a chain-level lifting of the map  $\gamma$  of (4.10). Under the identification (4.15), the map  $\gamma$  is induced by the natural map of complexes

$$(4.17) \quad \tilde{\gamma} : \pi_p^*\mathbb{K}_{\bullet} \rightarrow k$$

given by the pullback  $\pi_p^*(\kappa_0)$  of the map  $\kappa_0$  of (3.8) in homological degree 0.

The functor  $\pi_{pb}$  of (4.12) is exact, so that any filtration  $F^{\bullet}$  on a complex  $E_{\bullet}$  in  $\text{Fun}(\Lambda_p, R)$  induces a filtration

$$(4.18) \quad F^i \pi_{pb}E_{\bullet} = \pi_{pb}F^i E_{\bullet} \subset \pi_{pb}E_{\bullet}, \quad i \in \mathbb{Z}$$

on  $\pi_{pb}E_{\bullet}$ . If  $F^{\bullet}$  was termwise-split, then this induced filtration is also termwise-split, and if we equip  $cp_{\bullet}(E_{\bullet})$  with the standard filtration of Definition 3.4, the isomorphism (4.13) is a filtered isomorphism. Analogously, if we equip the product  $E_{\bullet} \otimes \pi_p^*\mathbb{K}_{\bullet}$  with the filtration induced by  $F^{\bullet}$ , and consider the corresponding standard filtration on  $cp_{\bullet}(E_{\bullet} \otimes \pi_p^*\mathbb{K}_{\bullet})$ , the identification (4.15) becomes a filtered isomorphism, and the map (4.16) is a filtered map.

**Lemma 4.7.** *Assume that  $pR = 0$ . Then for any complex  $E_{\bullet}$  in the category  $\text{Fun}(\Lambda_p, R)$ , we have a natural complex  $\tilde{cp}_{\bullet}(E_{\bullet})$  and a map*

$$\delta : \tilde{cp}_{\bullet}(E_{\bullet}) \rightarrow cph_{\bullet}(E_{\bullet})$$

*such that  $\gamma \circ \delta : \tilde{cp}_{\bullet}(E_{\bullet}) \rightarrow cp_{\bullet}(E_{\bullet})$  is a quasiisomorphism. Both  $\tilde{cp}_{\bullet}(E_{\bullet})$  and  $\delta$  are functorial in  $E_{\bullet}$ . Moreover, if  $E_{\bullet}$  carries a termwise-split filtration, then  $\tilde{cp}_{\bullet}(E_{\bullet})$  carries a functorial standard filtration such that  $\delta$  is a filtered map, and  $\gamma \circ \delta$  is a filtered quasiisomorphism.*

*Proof.* Since  $p$  annihilates  $R$ , we may replace the objects  $\mathbb{K}_0, \mathbb{K}_1$  in (2.7) and (3.8) with their reductions modulo  $p$ , and take the tensor product in (2.9) over  $k = \mathbb{Z}/p\mathbb{Z}$ . Then (3.8) provides an exact sequence

$$0 \longrightarrow k \xrightarrow{\kappa_1} \mathbb{K}_1 \longrightarrow \mathbb{K}_0 \xrightarrow{\kappa_0} k \longrightarrow 0$$

in  $\text{Fun}(\Lambda, k)$  that represents by Yoneda the generator  $u$  of the cohomology algebra  $H^\bullet(\Lambda, k) \cong k[u]$ . Since  $\pi_p^* u = 0$ , the sequence

$$0 \longrightarrow k \xrightarrow{\kappa_1} \pi_p^* \mathbb{K}_1 \longrightarrow \pi_p^* \mathbb{K}_0 \xrightarrow{\kappa_0} k \longrightarrow 0$$

represents the trivial class. Therefore by the standard criterion, there exists an object  $\mathbb{K}_{01} \in \text{Fun}(\Lambda_p, k)$  with a three-step filtration  $w^i \mathbb{K}_{01}$ ,  $i = 0, 1, 2$ , such that  $w^2 \mathbb{K}_{01} \cong k$ ,  $w^1 \mathbb{K}_{01} \cong \pi_p^* \mathbb{K}_1$ ,  $\mathbb{K}_{01}/w^2 \mathbb{K}_{01} \cong \pi_p^* \mathbb{K}_0$ ,  $\mathbb{K}_{01}/w^1 \mathbb{K}_{01} \cong k$ . Equivalently, denote by  $\tilde{\mathbb{K}}_\bullet$  the complex with terms  $\tilde{\mathbb{K}}_1 = \pi_p^* \mathbb{K}_1$ ,  $\tilde{\mathbb{K}}_0 = \mathbb{K}_{01}$ , and the differential given by the embedding  $\pi_p^* \mathbb{K}_1 \cong w^1 \mathbb{K}_{01} \hookrightarrow \mathbb{K}_{01}$ . Then the projection  $\mathbb{K}_{01} \rightarrow \mathbb{K}_0$  induces a map of complexes

$$(4.19) \quad \tilde{\delta} : \tilde{\mathbb{K}}_\bullet \rightarrow \pi_p^* \mathbb{K}_\bullet,$$

and its composition  $\tilde{\gamma} \circ \tilde{\delta} : \tilde{\mathbb{K}}_\bullet \rightarrow k$  with the map (4.17) is a surjective map of complexes with contractible kernel. To prove the lemma, it remains to denote

$$(4.20) \quad \tilde{E}_\bullet = E_\bullet \otimes_k \tilde{\mathbb{K}}_\bullet$$

and let  $\tilde{c}p_\bullet(E_\bullet) = cp_\bullet(\tilde{E}_\bullet)$ ,  $\tilde{\delta} = cp_\bullet(\text{id} \otimes \tilde{\delta})$ . Indeed,  $\tilde{E}_\bullet \rightarrow E_\bullet = \text{id} \otimes (\tilde{\gamma} \circ \tilde{\delta})$  is also a surjective map of complexes with contractible kernel, hence a strong quasiisomorphism in the sense of Definition 2.8, and therefore the map  $\delta \circ \gamma = cp_\bullet(\text{id} \otimes (\tilde{\delta} \circ \tilde{\gamma}))$  is a quasiisomorphism. The same is true in presence of filtrations.  $\square$

We note that since the map  $\delta$  provided by Lemma 4.7 is compatible with filtrations, it automatically has a completed counterpart. Namely, let  $\overline{CPH}_\bullet^{[p]}(E_\bullet)$  be the completion of the complex  $cph_\bullet(E_\bullet)$  with respect to the standard filtration corresponding to the  $p$ -th rescaling of the stupid filtration on  $E_\bullet$ . Then the map  $\gamma$  of (4.16) gives by completion a natural map

$$\gamma : \overline{CPH}_\bullet^{[p]}(E_\bullet) \rightarrow \overline{CP}_\bullet^{[p]}(E_\bullet),$$

where  $\overline{CP}_\bullet^{[p]}(E_\bullet)$  is as in Definition 4.3, and map  $\delta$  of Lemma 4.7 extends to a map of complete complexes such that  $\delta \circ \gamma$  is a quasiisomorphism.

Analogously, denote  $CPH_*(E_*) = CP_*(E_* \otimes \pi_p^* \mathbb{K}_*)$ . Then we have natural maps

$$CP_*(\tilde{E}_*) \xrightarrow{\delta} CPH_*(E_*) \xrightarrow{\gamma} CP_*(E_*)$$

whose completion is a quasiisomorphism. Moreover, Lemma 4.7 has an obvious counterpart for the restricted periodic complexes  $CP_*^f(E_*)$ ,  $\overline{CP}_*^f(E_*)$ . Namely, denote by

$$(4.21) \quad \tilde{\pi}_{pb} E_* = \text{Per}(\pi_{p!} \mathbb{K}_*(E_*)), \quad \check{\pi}_{pb} E_* = \overline{\text{Per}}(\pi_{p!} \mathbb{K}_*(E_*))$$

the periodic and co-periodic expansions of the mixed complex  $\pi_{p!}(\mathbb{K}_*(E_*))$ , and let

$$\begin{aligned} CPH_*^f(E_*) &= CH_*(\tilde{\pi}_{pb} E_*) \cong CP_*^f(E_* \otimes \pi_p^* \mathbb{K}_*), \\ \overline{CPH}_*^f(E_*) &= CH_*(\check{\pi}_{pb} E_*) \cong \overline{CP}_*^f(E_* \otimes \pi_p^* \mathbb{K}_*). \end{aligned}$$

Then we have natural maps

$$\begin{aligned} \gamma : CPH_*^f(E_*) &\rightarrow CP_*^f(E_*) \cong cc_*(\tilde{\pi}_{pb} E_*), \\ \gamma : \overline{CPH}_*^f(E_*) &\rightarrow \overline{CP}_*^f(E_*) \cong cc_*(\check{\pi}_{pb} E_*), \end{aligned}$$

and the same argument as in Lemma 4.7 provides maps  $\delta$  such that  $\gamma \circ \delta$  is a quasiisomorphism.

## 5 Computational tools.

**5.1 Conjugate spectral sequence.** As in Subsection 4.2 and Subsection 4.3, assume that the base ring  $R$  is annihilated by an odd prime  $p$ , and let  $k = \mathbb{Z}/p\mathbb{Z}$ . Then for any complex  $E_*$  in  $\text{Fun}(\Lambda, R)$ , we have the filtered quasiisomorphism  $\nu_p$  of Proposition 4.4 and its completed version (4.5). To study the complex  $\overline{CP}^{[p]}(i_p^* E_*)$ , we will use the truncation functors  $\tau^\bullet$ ,  $\beta^\bullet$  of Subsection 1.3.

**Definition 5.1.** For any termwise-split filtered complex  $E_*$  in  $\text{Fun}(\Lambda_p, R)$ , the filtrations  $V^\bullet$  and  $W^\bullet$  on  $cp_*(E_*)$  are given by

$$(5.1) \quad V^n = \tau^{2n-1} cp_*(E_*), \quad W^n = \beta^{2n-2} cp_*(E_*), \quad n \in \mathbb{Z},$$

where we equip  $cp_*(E_*)$  with the standard filtration induced by the filtration on  $E_*$ . The filtration  $V^\bullet$  is called the *conjugate filtration*.

Note that both filtrations are 2-periodic – for every integer  $n$ , the periodicity endomorphism  $u : cp_\bullet(E_\bullet) \rightarrow cp_\bullet(E_\bullet)[2]$  induces isomorphisms

$$(5.2) \quad u : V^{n+1}cp_\bullet(E_\bullet) \cong V^n cp_\bullet(E_\bullet)[2], \quad u : W^{n+1}cp_\bullet(E_\bullet) \cong W^n cp_\bullet(E_\bullet)[2],$$

where  $[2]$  stands for the cohomological shift. By Lemma 1.8, we have  $V^{n+1}cp_\bullet(E_\bullet) \subset W^n cp_\bullet(E_\bullet) \subset V^n cp_\bullet(E_\bullet)$ , and the first embedding is a quasi-isomorphism. The conjugate filtration  $V^\bullet$  is by definition the rescaling by 2 and shift by 1 of the filtration  $\tau^\bullet$ . In particular,  $V^\bullet$  and  $\tau^\bullet$  are commensurable, so that by Lemma 1.8, the conjugate filtration is commensurable with the standard filtration. Thus if we equip a complex  $E_\bullet$  in  $\text{Fun}(\Lambda_p, R)$  with the  $p$ -th rescaling of the stupid filtration, the completion of  $cp_\bullet(E_\bullet)$  with respect to the conjugate filtration coincides with the complex  $\overline{CP}^{[p]}(\tilde{E}_\bullet)$  of Definition 4.3. By virtue of the quasiisomorphism (4.5), we then have a spectral sequence

$$(5.3) \quad H_\bullet(\text{gr}_V^\bullet(cp_\bullet(i_p^* E_\bullet^{[p]}))) \Rightarrow \overline{HP}^{[p]}_\bullet(i_p^* E_\bullet^{[p]}) = \overline{HP}_\bullet(E_\bullet)$$

for any complex  $E_\bullet$  in  $\text{Fun}(\Lambda, R)$ . As it turns out, this spectral sequence is quite useful, because under some assumptions on  $E_\bullet$ , one can find a rather effective description of its initial term.

To do this, we need to recall some material from [Ka3]. The cohomology  $H^\bullet(\mathbb{Z}/p\mathbb{Z}, k)$  with coefficients in  $k = \mathbb{Z}/p\mathbb{Z}$  is the graded-commutative algebra given by

$$(5.4) \quad H^\bullet(\mathbb{Z}/p\mathbb{Z}, k) = k[u]\langle \varepsilon \rangle,$$

where  $u$  is a generator of degree 2 and  $\varepsilon$  is a generator of degree 1. The generator  $\varepsilon$  gives an extension  $\tilde{k}$  of the trivial  $k[\mathbb{Z}/p\mathbb{Z}]$ -module  $k$  by itself, so that for any  $R[\mathbb{Z}/p\mathbb{Z}]$ -module  $E$ , we have a functorial short exact sequence

$$(5.5) \quad 0 \longrightarrow E \longrightarrow \tilde{E} \longrightarrow E \longrightarrow 0,$$

where we let  $\tilde{E} = E \otimes_k \tilde{k}$ . Taking the Tate homology complex  $\check{C}_\bullet(-)$  of (2.11), we obtain a short exact sequence of complexes

$$0 \longrightarrow \check{C}_\bullet(\mathbb{Z}/p\mathbb{Z}, E) \longrightarrow \check{C}_\bullet(\mathbb{Z}/p\mathbb{Z}, \tilde{E}) \longrightarrow \check{C}_\bullet(\mathbb{Z}/p\mathbb{Z}, E) \longrightarrow 0.$$

This defines a distinguished triangle in  $\mathcal{DF}(R)$ , so that we have a connecting differential  $\check{C}_\bullet(\mathbb{Z}/p\mathbb{Z}, E) \rightarrow \check{C}_\bullet(\mathbb{Z}/p\mathbb{Z}, E)[1]$  and the corresponding maps

$$(5.6) \quad \varepsilon_i : \check{H}_i(\mathbb{Z}/p\mathbb{Z}, E) \rightarrow \check{H}_{i-1}(\mathbb{Z}/p\mathbb{Z}, E)$$

for any integer  $i$ . In effect, the cohomology algebra  $H^\bullet(\mathbb{Z}/p\mathbb{Z}, k)$  acts on  $\check{H}_\bullet(\mathbb{Z}/p\mathbb{Z}, E)$ , and the maps  $\varepsilon_i$  give the action of the generator  $\varepsilon$ . The generator  $u$  acts by the 2-shift in the periodic complex  $\check{C}_\bullet(\mathbb{Z}/p\mathbb{Z}, E)$ , so that  $\varepsilon_i = \varepsilon_{i+2}$  for any  $i$ . Thus effectively, we only have two maps  $\varepsilon_{\text{odd}}$  and  $\varepsilon_{\text{even}}$ , depending on the parity of  $i$ . An  $R[\mathbb{Z}/p\mathbb{Z}]$ -module  $E$  is called *tight* if the map  $\varepsilon_{\text{odd}}$  is an isomorphism. Since  $\varepsilon^2 = 0$ , we have  $\varepsilon_{\text{odd}} \circ \varepsilon_{\text{even}} = 0$ ; thus for a tight module, we automatically have  $\varepsilon_{\text{even}} = 0$ .

**Definition 5.2.** A complex  $E_\bullet$  of  $R[\mathbb{Z}/p\mathbb{Z}]$ -modules is *tight* if  $E_i$  is a tight  $R[\mathbb{Z}/p\mathbb{Z}]$ -module for any  $i$ , and  $\mathsf{l}(E_i) = 0$  unless  $i$  divides  $p$ .

Now assume given a filtered complex  $E_\bullet$  of  $R[\mathbb{Z}/p\mathbb{Z}]$ -modules, and consider the Tate complex  $\check{C}_\bullet(\mathbb{Z}/p\mathbb{Z}, E_\bullet)$  of (2.12). Equip it with the filtration  $F^\bullet$  induced by the filtration on  $E_\bullet$ , and for any integer  $i$ , denote

$$\tilde{H}_i(\mathbb{Z}/p\mathbb{Z}, E_\bullet) \cong H_i(\check{C}_\bullet(\mathbb{Z}/p\mathbb{Z}, E_\bullet)),$$

where  $H_i(-)$  is as in (1.9). Then again, for any integer  $i$ , (5.5) induces a natural map

$$(5.7) \quad \varepsilon_i : \tilde{H}_i(\mathbb{Z}/p\mathbb{Z}, E_\bullet) \rightarrow \tilde{H}_{i-1}(\mathbb{Z}/p\mathbb{Z}, E_\bullet)[1],$$

a generalization of (5.6). As before,  $\varepsilon_i$  only depends on the parity of  $i$ , so that effectively, we only have two maps  $\varepsilon_{\text{odd}}$  and  $\varepsilon_{\text{even}}$ .

**Lemma 5.3.** Assume given a complex  $E_\bullet$  of  $R[\mathbb{Z}/p\mathbb{Z}]$ -modules tight in the sense of Definition 5.2, and equip it with the  $p$ -th rescaling  $F_{[p]}^\bullet$  of the stupid filtration  $F^\bullet$ . Then  $\varepsilon_{\text{even}} = 0$ , and  $\varepsilon_{\text{odd}}$  is an isomorphism.

*Proof.* Immediately follows from Lemma 1.8 (note that since  $p$  is odd by assumption,  $(p-1)i$  is even for any  $i$ ).  $\square$

By virtue of Lemma 5.3, for any tight complex  $E_\bullet$  of  $R[\mathbb{Z}/p\mathbb{Z}]$ -modules, the complexes  $\tilde{H}_i(\mathbb{Z}/p\mathbb{Z}, E_\bullet)[-i]$  for all integers  $i$  are canonically identified. We denote this complex by  $\mathsf{l}(E_\bullet)$ . By Lemma 1.8, the natural filtration on  $\mathsf{l}(E_\bullet)$  coincides with the stupid filtration, and we have

$$(5.8) \quad \mathsf{l}(E_\bullet)_n = \mathsf{l}(E_{pn})$$

for any integer  $n$ .

Assume now given a complex  $E_\bullet$  in  $\text{Fun}(\Lambda_p, R)$ . Then for any object  $[n] \in \Lambda_p$ ,  $E_\bullet([n])$  is naturally a complex of  $R[\mathbb{Z}/p\mathbb{Z}]$ -modules via the embedding  $\mathbb{Z}/p\mathbb{Z} \subset \mathbb{Z}/pn\mathbb{Z} = \text{Aut}([n])$ .

**Definition 5.4.** A complex  $E_\bullet$  in  $\text{Fun}(\Lambda_p, R)$  is *tight* if  $E_\bullet([n])$  is a tight complex of  $R[\mathbb{Z}/p\mathbb{Z}]$ -modules for any  $[n] \in \Lambda_p$ . A complex  $E_\bullet$  in  $\text{Fun}(\Lambda, R)$  is *p-adapted* if  $i_p^* E_\bullet$  is tight.

**Definition 5.5.** For any tight complex  $E_\bullet$  in  $\text{Fun}(\Lambda_p, R)$ , the complex  $l(E_\bullet)$  in  $\text{Fun}(\Lambda, R)$  is given by

$$l(E_\bullet)([n]) = l(E_\bullet([n]))$$

for every  $[n] \in \Lambda$ .

Note that for every  $n \geq 1$ , we have a base change isomorphism

$$(5.9) \quad \pi_{pb} E_\bullet([n]) \cong \tilde{C}_\bullet(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/pn\mathbb{Z}, E_\bullet([n])),$$

where the right-hand side is the extended version (2.15) of the Tate homology complex  $\tilde{C}_\bullet(\mathbb{Z}/p\mathbb{Z}, -)$ . Since for any  $n$ , the complex  $\tilde{C}_\bullet(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/pn\mathbb{Z}, -)$  is canonically chain-homotopy equivalent to the usual Tate homology complex  $\tilde{C}_\bullet(\mathbb{Z}/p\mathbb{Z}, -)$ , the isomorphism (5.9) provides a natural identification

$$l(E_\bullet) \cong H_i(\pi_{pb} E_\bullet^{[p]})[-i]$$

for any integer  $i$ , where  $E_\bullet^{[p]}$  is  $E_\bullet$  equipped with the  $p$ -th rescaling of the stupid filtration, and  $H_i(-)$  is the truncation functor (1.9) in the category of termwise-split filtered complexes in  $\text{Fun}(\Lambda, R)$ . By abuse of notation, for any complex  $E_\bullet$  in  $\text{Fun}(\Lambda, R)$ , we will denote

$$l(E_\bullet) = l(i_p^* E_\bullet).$$

With these definitions, the main result concerning the spectral sequence (5.3) is the following.

**Proposition 5.6.** *Assume that  $pR = 0$  for an odd prime  $p$ . Then for any tight complex  $E_\bullet$  in  $\text{Fun}(\Lambda_p, R)$ , we have a natural quasiisomorphism*

$$(5.10) \quad \text{gr}_V^0(cp_\bullet(E_\bullet^{[p]})) \cong CH_\bullet(l(E_\bullet))$$

*functorial in  $E_\bullet$ , so that for any  $p$ -adapted complex  $E_\bullet$  in  $\text{Fun}(\Lambda, R)$ , (5.3) induces a functorial spectral sequence*

$$(5.11) \quad HH_\bullet(l(E_\bullet))((u^{-1})) \Rightarrow \overline{HP}_\bullet(E_\bullet).$$

Note that by virtue of the periodicity isomorphism (5.2), the identification (5.10) describes the whole associated graded quotient  $\text{gr}_V^\bullet cp_\bullet(E_\bullet)$ . In (5.11), we use the same shorthand notation as in Definition 2.3. We will call the spectral sequence (5.11) the *conjugate spectral sequence* for the tight complex  $E_\bullet$ .

**5.2 Localization.** To prove Proposition 5.6, we first need to localize the conjugate filtration onto the category  $\Lambda$ . For any complex  $E_\bullet$  in  $\text{Fun}(\Lambda, R)$  equipped with a termwise-split filtration  $F^\bullet$ , we denote by  $V^\bullet E_\bullet$  and  $W^\bullet E_\bullet$  the filtrations given by

$$(5.12) \quad V^n E_\bullet = \tau^{2n} E_\bullet, \quad W^n E_\bullet = \beta^{2n-1} E_\bullet, \quad n \in \mathbb{Z},$$

where  $\tau^\bullet$  and  $\beta^\bullet$  are as in (1.8). By Lemma 1.8, for any  $E_\bullet$ , we have  $V^{n+1} E_\bullet \subset W^n E_\bullet \subset V^n E_\bullet$ , and the first embedding is an isomorphism.

In particular, assume given a complex  $E_\bullet$  in the category  $\text{Fun}(\Lambda_p, R)$ , equip it with the  $p$ -th rescaling of the stupid filtration, and to simplify notation, denote by  $E_\bullet^\flat = \pi_{pb} E_\bullet^{[p]}$  the complex (4.12) with the filtration (4.18). Then we have filtrations  $V^\bullet, W^\bullet$  on  $E_\bullet^\flat$ , and both filtrations are periodic in the same sense as (5.2). Analogously, let  $\tilde{E}_\bullet^\flat = \pi_{pb} \tilde{E}_\bullet$ , where  $\tilde{E}_\bullet$  is the complex (4.20) with the filtration induced from  $E_\bullet$ . Then  $\tilde{E}_\bullet^\flat$  also carries periodic filtrations  $V^\bullet, W^\bullet$ . By virtue of the identification (4.14), the map (4.19) induces a map

$$(5.13) \quad \delta^\flat : \tilde{E}_\bullet^\flat \rightarrow E_\bullet^\flat \otimes \mathbb{K}_\bullet.$$

**Lemma 5.7.** *Assume that a complex  $E_\bullet$  in  $\text{Fun}(\Lambda_p, R)$  is tight in the sense of Definition 5.4. Then the following is true.*

- (i) *For any integer  $n$ , the map  $\delta^\flat$  of (5.13) sends  $V^n \tilde{E}_\bullet^\flat \subset \tilde{E}_\bullet^\flat$  into  $W^n E_\bullet^\flat \otimes \mathbb{K}_\bullet \subset E_\bullet^\flat \otimes \mathbb{K}_\bullet$ .*
- (ii) *Moreover, denote  $\mathbb{I}_\bullet = \text{gr}_\beta^{-1} E_\bullet^\flat$ , and let  $\rho : W^0 E_\bullet^\flat = \beta^{-1} E_\bullet^\flat \rightarrow \mathbb{I}_\bullet$  be the natural projection. Then the composition map*

$$\text{gr}_V^0 \tilde{E}_\bullet^\flat \xrightarrow{\delta^\flat} \text{gr}_W^0 E_\bullet^\flat \otimes \mathbb{K}_\bullet \xrightarrow{\rho \otimes \text{id}} \mathbb{I}_\bullet \otimes \mathbb{K}_\bullet$$

*is a quasiisomorphism.*

*Proof.* Since by definition, the filtration on  $E_\bullet$  is termwise-split, both claims commute with passing to the associated graded quotients. Therefore we may assume right away that  $E_\bullet$  is concentrated in a single filtered degree, say 0. Moreover, by periodicity, it suffices to prove (i) for  $n = 0$ . We have  $V^0 = \tau^0$ ,  $W^0 = \beta^{-1}$ , and

$$\tau^0(E_\bullet^\flat \otimes \mathbb{K}_\bullet) / ((\beta^{-1} E_\bullet^\flat \otimes \mathbb{K}_\bullet) \cap \tau^0(E_\bullet^\flat \otimes \mathbb{K}_\bullet)) \cong H_{-1}(E_\bullet^\flat) \otimes H_1(\mathbb{K}_\bullet).$$

Since  $H_1(\mathbb{K}_\bullet) \cong \mathbb{Z}$ , we have  $H_{-1}(E_\bullet^\flat) \otimes H_1(\mathbb{K}_\bullet) \cong H_{-1}(E_\bullet^\flat)$ , and the map

$$\tau^0 \tilde{E}_\bullet^\flat \xrightarrow{\delta^\flat} \tau^0(E_\bullet^\flat \otimes \mathbb{K}_\bullet) \longrightarrow H_{-1}(E_\bullet^\flat) \otimes H_1(\mathbb{K}_\bullet) \cong H_{-1}(E_\bullet^\flat)$$

factors through a map

$$(5.14) \quad H_0(\tilde{E}_\bullet^\flat) = \tau^0 \tilde{E}_\bullet^\flat / \beta^0 \tilde{E}_\bullet^\flat \rightarrow H_{-1}(E_\bullet^\flat).$$

We have to show that this map is equal to 0. This claim is local with respect to  $\Lambda$ , so that it suffices to prove it after evaluation at an arbitrary object  $[n] \in \Lambda$ . We have

$$\tilde{E}_\bullet^\flat([n]) \cong \tilde{C}_\bullet(\mathbb{Z}/p\mathbb{Z}, M_\bullet \otimes_k K_\bullet), \quad E_\bullet^\flat([n]) \cong \tilde{C}_\bullet(\mathbb{Z}/p\mathbb{Z}, M_\bullet),$$

where we denote  $M_\bullet = E_\bullet([n])$ ,  $K_\bullet = \tilde{\mathbb{K}}_\bullet([n])$ , and (5.14) evaluates to a map

$$(5.15) \quad \check{H}_0(\mathbb{Z}/p\mathbb{Z}, M \otimes K_\bullet) \rightarrow \check{H}_{-1}(\mathbb{Z}/p\mathbb{Z}, M),$$

where  $M = H_0(M_\bullet)$  stand for the only non-trivial homology group of the complex  $M_\bullet$ .

However, since the  $\mathbb{Z}/p\mathbb{Z}$ -action on  $K_1 = \pi_p^* \mathbb{K}_1([n])$  is trivial, the embedding  $\kappa_1 : k \rightarrow K_1$  splits as a map of  $k[\mathbb{Z}/p\mathbb{Z}]$ -modules. Choosing such a splitting gives a quasiisomorphism between  $K_\bullet$  and the complex  $\overline{K}_\bullet$  with terms  $\overline{K}_1 = k$ ,  $\overline{K}_0 = \tilde{k}$ , and the differential given by the embedding  $k \rightarrow \tilde{k}$  of (5.5). Then in defining the map (5.15), we may replace  $K_\bullet$  with  $\overline{K}_\bullet$ , and the map becomes precisely the map  $\varepsilon_{\text{even}}$  for the  $k[\mathbb{Z}/p\mathbb{Z}]$ -module  $M$ . Since  $M$  is tight by assumption,  $\varepsilon_{\text{even}} = 0$ . This proves (i).

The argument for (ii) is similar — the non-trivial part is to check that the natural map

$$H_1(\tilde{E}_\bullet^\flat) \rightarrow H_0(\mathbf{l}_\bullet) \otimes H_1(\mathbb{K}_\bullet) \cong H_0(E_\bullet^\flat)$$

induced by  $\delta^\flat$  is an isomorphism, this is a local fact, and after evaluation at  $[n] \in \Lambda$  and choosing a quasiisomorphism  $K_\bullet \cong \overline{K}_\bullet$ , the map becomes the map  $\varepsilon_{\text{odd}}$  for the tight  $k[\mathbb{Z}/p\mathbb{Z}]$ -module  $M$ . We leave the details to the reader.  $\square$

**Remark 5.8.** Lemma 5.7 (ii) is a strengthening of [Ka3, Lemma 3.6]. The proof is also essentially the same; the only difference is that instead of the complex  $\tilde{\mathbb{K}}_\bullet$ , [Ka3] uses an arbitrary resolution of the constant functor  $k$  by objects in  $\text{Fun}(\Lambda_p, R)$  acyclic for the functor  $\pi_{p!}$ .

Now note that by Lemma 1.9, for any integer  $n$ , the inclusion  $V^n \tilde{E}_\bullet^\flat \subset \tilde{E}_\bullet^\flat$  induces a surjective map

$$\xi : cc_\bullet(V^n \tilde{E}_\bullet^\flat) \rightarrow V^n cc_\bullet(\tilde{E}_\bullet^\flat) \cong V^n cp_\bullet(\tilde{E}_\bullet^\flat),$$



where  $V^n$  in the right-hand side is the conjugate filtration of Definition 5.1, and the shift by 1 between (5.1) and (5.12) compensates for the shift by one in Definition 3.4. Composing  $\xi$  with the map  $\alpha$  of (3.12), we obtain a natural map

$$(5.16) \quad CC_{\bullet}(V^n \tilde{E}_{\bullet}^b) \rightarrow V^n cp_{\bullet}(\tilde{E}_{\bullet}).$$

Since  $CC_{\bullet}(-)$  is an exact functor, it induces a map

$$(5.17) \quad CC_{\bullet}(V^{[n,m]} \tilde{E}_{\bullet}^b) \rightarrow V^{[n,m]}(\tilde{E}_{\bullet})$$

for any integer  $m \geq n$ .

**Lemma 5.9.** *For any tight filtered complex  $E_{\bullet}$  in  $\text{Fun}(\Lambda_p, R)$ , and any integer  $n \leq m$ , the map (5.17) is a quasiisomorphism.*

*Proof.* By induction, it suffices to consider the case  $m = n + 1$ , and then by periodicity, it suffices to prove that the map

$$CC_{\bullet}(\text{gr}_V^0 \tilde{E}_{\bullet}^b) \rightarrow \text{gr}_V^0 cp_{\bullet}(\tilde{E}_{\bullet})$$

induced by (5.16) is a quasiisomorphism. Lemma 4.7 provides maps

$$cp_{\bullet}(\tilde{E}_{\bullet}) = \tilde{cp}_{\bullet}(E_{\bullet}) \xrightarrow{\delta} cph_{\bullet}(E^{\bullet}) \xrightarrow{\gamma} cp_{\bullet}(E_{\bullet})$$

whose composition is a filtered quasiisomorphism. By definition, we have  $cph_{\bullet}(E_{\bullet}) \cong CH_{\bullet}(E_{\bullet}^b)$ , and since  $CH_{\bullet}(-)$  is an exact functor, the natural map  $\iota : CH_{\bullet}(W^0 E_{\bullet}^b) \rightarrow CH_{\bullet}(E_{\bullet}^b) = cph_{\bullet}(E_{\bullet})$  is injective. By Lemma 5.7 (i), we then have a commutative square of complexes

$$\begin{array}{ccc} cc_{\bullet}(V^0 \tilde{E}_{\bullet}^b) & \xrightarrow{cc_{\bullet}(\delta^b)} & CH_{\bullet}(W^0 E_{\bullet}^b) \\ \xi \downarrow & & \downarrow \iota \\ V^0 cc_{\bullet}(\tilde{E}_{\bullet}^b) & \xrightarrow{\delta} & cph_{\bullet}(E_{\bullet}). \end{array}$$

Since  $\xi$  is surjective and  $\iota$  is injective, there exists a map  $\bar{\delta} : V^0 cc_{\bullet}(\tilde{E}_{\bullet}^b) \rightarrow CH_{\bullet}(W^0 E_{\bullet}^b)$  such that  $cc_{\bullet}(\delta^b) = \bar{\delta} \circ \xi$  and  $\delta = \iota \circ \bar{\delta}$ . Now note that the map  $\tilde{\gamma}$  of (4.17) induces a map  $\gamma^b : E_{\bullet}^b \otimes \mathbb{K}_{\bullet} \rightarrow E_{\bullet}^b$  such that  $\gamma^b \circ \delta^b$  is a filtered quasiisomorphism, so that altogether, we have a commutative diagram

$$\begin{array}{ccccc} CC_{\bullet}(\text{gr}_V^0 \tilde{E}_{\bullet}^b) & \xrightarrow{CC_{\bullet}(\delta^b)} & CC_{\bullet}(\text{gr}_W^0 E_{\bullet}^b \otimes \mathbb{K}_{\bullet}) & \xrightarrow{CC_{\bullet}(\gamma^b)} & CC_{\bullet}(\text{gr}_W^0 E_{\bullet}^b) \\ \downarrow & & \alpha \downarrow & & \downarrow \\ \text{gr}_V^0 cp_{\bullet}(\tilde{E}_{\bullet}) & \xrightarrow{\bar{\delta}} & CH_{\bullet}(\text{gr}_W^0 E_{\bullet}^b) & \xrightarrow{\gamma \circ \iota} & \text{gr}_W^0 cp_{\bullet}(E_{\bullet}). \end{array}$$

The composition  $\gamma \circ \iota \circ \bar{\delta} = \gamma \circ \delta$  is a quasiisomorphism. Since  $CC_{\bullet}(-)$  is an exact functor,  $CC_{\bullet}(\gamma^b) \circ CC_{\bullet}(\delta^b) = CC_{\bullet}(\gamma^b \circ \delta^b)$  is also a quasiisomorphism, and the map  $\alpha$  is a quasiisomorphism by Lemma 3.11. Therefore the remaining vertical arrows are quasiisomorphisms, and we are done.  $\square$

*Proof of Proposition 5.6.* Assume given a tight complex  $E_{\bullet}$  in  $\text{Fun}(\Lambda_p, R)$ , equip it with the  $p$ -th rescaling of the stupid filtration, and keep the notation introduced earlier in this Subsection. Then Lemma 5.7 and Lemma 3.11 provide a quasiisomorphism

$$CC_{\bullet}(\text{gr}_V^0 \tilde{E}_{\bullet}^b) \cong CC_{\bullet}(\mathbb{I}_{\bullet} \otimes \mathbb{K}_{\bullet}) \cong CH_{\bullet}(\mathbb{I}_{\bullet}),$$

and Lemma 5.9 further identifies this with  $\text{gr}_V^0 cp_{\bullet}(\tilde{E}_{\bullet})$ . It remains to notice that since the embedding  $\tau^0 E_{\bullet}^b \subset \beta^{-1} E_{\bullet}^b$  is a quasiisomorphism, the embedding  $\mathbb{I}(E_{\bullet}) \subset \mathbb{I}_{\bullet}$  is a quasiisomorphism, so that  $CH_{\bullet}(\mathbb{I}_{\bullet})$  is quasiisomorphic to  $CH_{\bullet}(\mathbb{I}(E_{\bullet}))$ , and moreover, since  $E_{\bullet}$  is quasiisomorphic to  $\tilde{E}_{\bullet}$  as a filtered complex,  $\text{gr}_V^0 cp_{\bullet}(\tilde{E}_{\bullet})$  is quasiisomorphic to  $\text{gr}_V^0 cp_{\bullet}(E_{\bullet})$ .  $\square$

**5.3 Comparison maps.** We now turn to the other functorial complexes in (3.23) and the comparison maps  $r, R, l, L$  between them. Keep the assumption  $pR = 0$ ,  $p$  an odd prime. Recall that for any complex  $E_{\bullet}$  in  $\text{Fun}(\Lambda_p, R)$ , we have the complex  $\pi_{pb} E_{\bullet}$  of (4.12) and its completed versions (4.21).

**Definition 5.10.** A complex  $E_{\bullet}$  in  $\text{Fun}(\Lambda_p, R)$  is *locally bounded from below* resp. *locally strongly bounded from above* if for any  $[n] \in \Lambda_p$ , the complex  $E_{\bullet}([n])$  of  $R[\mathbb{Z}/pn\mathbb{Z}]$ -modules is bounded from below resp. strongly bounded from above in the sense of Definition 2.9.

**Lemma 5.11.** Assume given a complex  $E_{\bullet}$  in  $\text{Fun}(\Lambda_p, R)$ . Then the natural map  $\alpha : CC_{\bullet}(\pi_{pb} E_{\bullet}) \rightarrow cp_{\bullet}(E_{\bullet})$  induced by the map (3.12) is a quasiisomorphism, and so are the natural maps

$$(5.18) \quad \alpha : CC_{\bullet}(\tilde{\pi}_{pb} E_{\bullet}) \rightarrow CP_{\bullet}^f(E_{\bullet}), \quad \alpha : CC_{\bullet}(\check{\pi}_{pb} E_{\bullet}) \rightarrow \overline{CP}_{\bullet}^f(E_{\bullet})$$

*Proof.* As in the proof of Lemma 5.9, Lemma 4.7 shows that to prove the first claim, it suffices to prove that the natural map

$$CC_{\bullet}(\pi_{pb} E_{\bullet} \otimes \mathbb{K}_{\bullet}) \rightarrow cph_{\bullet}(E_{\bullet}) \cong CH_{\bullet}(\pi_{pb} E_{\bullet})$$

is a quasiisomorphism. This immediately follows from Lemma 3.11. For the maps (5.18), use the counterpart of Lemma 4.7 for the restricted complexes, and again apply Lemma 3.11.  $\square$

**Corollary 5.12.** *Assume given a complex  $E_\bullet$  in  $\text{Fun}(\Lambda_p, R)$ . If  $E_\bullet$  is locally bounded from below, then the natural map  $l$  of (3.23) is a quasiisomorphism. If  $E_\bullet$  is locally strongly bounded from above, then the map  $r$  of (3.23) is a quasiisomorphism.*

*Proof.* By Lemma 5.11, the maps  $l$  and  $r$  are obtained by applying the exact functor  $CC_\bullet(-)$  to their local counterparts

$$l_{\flat} : \pi_{p\flat} E_\bullet \rightarrow \check{\pi}_{p\flat} E_\bullet, \quad r_{\flat} : \pi_{p\flat} E_\bullet \rightarrow \check{\pi}_{p\flat} E_\bullet.$$

Therefore both claims immediately follow from Lemma 2.10.  $\square$

The situation for  $L$  and  $R$ , the other two comparison maps in (3.23), is more difficult. In fact, we can only prove any useful statements in the special case of cyclic complexes coming from DG algebras; this we do later in Section 6. For now, we prepare the ground by axiomatizing the situation and proving some easy auxiliary results.

First of all, what we are really interested in are complexes  $E_\bullet$  in the category  $\text{Fun}(\Lambda, R)$ , but we study them by applying the quasiisomorphism of Proposition 4.4 and its completed versions (4.5), (4.9). Thus for a complex  $E_\bullet$  in  $\text{Fun}(\Lambda_p, R)$ , we need to consider the completion  $\overline{CP}^{[p]}(E_\bullet)$  of the complex  $cp_\bullet(E_\bullet)$  introduced in Definition 4.3.

**Lemma 5.13.** *For any complex  $E_\bullet$  in  $\text{Fun}(\Lambda_p, R)$ , there exists a functorial map  $L^{[p]} : \overline{CP}^f(E_\bullet) \rightarrow \overline{CP}^{[p]}(E_\bullet)$  such that  $L^{[p]} \circ l : cp_\bullet(E_\bullet) \rightarrow \overline{CP}^{[p]}(E_\bullet)$  is the completion map.*

*Proof.* Recall that by definition, we have

$$cp_\bullet(E_\bullet) = cc_\bullet(\text{per}(\mathbb{K}_\bullet(E_\bullet))) = \bigoplus_{i \geq 0} cc_i(\text{per}(\mathbb{K}_\bullet(E_{\bullet-i}))),$$

where the right-hand side is the decomposition (1.10). For any  $i \geq 0$ , the restriction of the standard filtration on  $cp_\bullet(E_\bullet^{[p]})$  to the summand  $cc_i(-)$  in this decomposition is a shift of the filtration induced by the  $p$ -th rescaling of the stupid filtration on  $E_\bullet$ . Therefore by Lemma 2.11, the completion of  $cc_i(-)$  with respect to this restricted filtration is precisely  $cc_i(\overline{\text{Per}}(\mathbb{K}_\bullet(E_{\bullet-i})))$ . Summing up over all  $i$ , we obtain a map

$$L^{[p]} : \bigoplus_{i \geq 0} cc_i(\overline{\text{Per}}(\mathbb{K}_\bullet(E_{\bullet-i}))) \rightarrow \overline{CP}^{[p]}(E_\bullet),$$

and again by (1.10), the left-hand is exactly  $\overline{CP}^f_\bullet(E_\bullet)$ .  $\square$

Next, we note that the embedding  $j_p : \Delta^o \rightarrow \Lambda_p$  extends to an embedding  $\tilde{j}_p : \Delta^o \times \mathbf{pt}_p \rightarrow \Lambda_p$ , where  $\mathbf{pt}_p$  is the groupoid with one object with automorphism group  $\mathbb{Z}/p\mathbb{Z}$ . Therefore the pullback functor  $j_p^*$  can be refined to a functor

$$(5.19) \quad \tilde{j}_p^* : \text{Fun}(\Lambda_p, R) \rightarrow \text{Fun}(\Delta^o \times \mathbf{pt}_p, R) \cong \text{Fun}(\Delta^o, R[\mathbb{Z}/p\mathbb{Z}]).$$

Moreover, if we denote by  $\pi_p : \Delta^o \times \mathbf{pt}_p \rightarrow \Delta^o$  the projection onto the first component, then we have a base change isomorphism  $\pi_{p!} \circ \tilde{j}_p^* \cong j^* \circ \pi_{p!}$ . For any complex  $E_\bullet$  in the category  $\text{Fun}(\Delta^o, R[\mathbb{Z}/p\mathbb{Z}])$ , we then denote by  $\mathbb{K}_\bullet(E_\bullet)$  the product  $E_\bullet \otimes \tilde{j}_p^* K_\bullet$ , and we define  $\pi_{pb} E_\bullet$ ,  $\tilde{\pi}_{pb} E_\bullet$ ,  $\check{\pi}_{pb} E_\bullet$  by (4.12) and (4.21). We let  $cph_\bullet(E_\bullet) = CH_\bullet(\pi_{pb} E_\bullet) = \text{per}(CH_\bullet(\pi_{p!} \mathbb{K}_\bullet(E_\bullet)))$  and

$$(5.20) \quad CPH^f_\bullet(E_\bullet) = CH_\bullet(\tilde{\pi}_{pb} E_\bullet), \quad \overline{CPH}^f_\bullet(E_\bullet) = CH_\bullet(\check{\pi}_{pb} E_\bullet).$$

We have  $\pi_{pb} \circ \tilde{j}_p^* \cong j^* \circ \pi_{pb}$ ,  $\tilde{\pi}_{pb} \circ \tilde{j}_p^* \cong j^* \circ \tilde{\pi}_{pb}$ ,  $\check{\pi}_{pb} \circ \tilde{j}_p^* \cong j^* \circ \check{\pi}_{pb}$ , so that for any complex  $E_\bullet$  in  $\text{Fun}(\Lambda_p, R)$ , we have a natural map

$$(5.21) \quad cph_\bullet(\tilde{j}_p^* E_\bullet) \rightarrow cph_\bullet(E_\bullet)$$

and natural maps

$$(5.22) \quad J^f : CPH^f_\bullet(\tilde{j}_p^* E_\bullet) \rightarrow CPH_\bullet(E_\bullet), \quad \bar{J}^f : \overline{CPH}^f_\bullet(\tilde{j}_p^* E_\bullet) \rightarrow \overline{CPH}_\bullet(E_\bullet).$$

Moreover, we let  $CPH_\bullet(E_\bullet) = \text{Per}(CH_\bullet(\pi_{p!} \mathbb{K}_\bullet(E_\bullet)))$ , and we denote by  $CPH^{[p]}_\bullet(E_\bullet)$  the completion of the filtered complex  $cph^f(E^{[p]}_\bullet)$ , where  $E^{[p]}_\bullet$  is  $E_\bullet$  equipped with the  $p$ -th rescaling of the stupid filtration, and  $cph^f(-)$  is the filtered extension (1.11) of the functor  $cph_\bullet(-)$ . Then for any complex  $E_\bullet$  in  $\text{Fun}(\Lambda_p, R)$ , the map (5.21) induces natural maps

$$(5.23) \quad J : CPH_\bullet(\tilde{j}_p^* E_\bullet) \rightarrow CPH_\bullet(E_\bullet), \quad \bar{J} : \overline{CPH}^{[p]}_\bullet(\tilde{j}_p^* E_\bullet) \rightarrow \overline{CPH}^{[p]}_\bullet(E_\bullet).$$

The same argument as in Lemma 5.13 shows that for any complex  $\tilde{E}_\bullet$  in  $\text{Fun}(\Delta^o, R[\mathbb{Z}/p\mathbb{Z}])$ , we have natural maps

$$(5.24) \quad R : CPH^f_\bullet(\tilde{E}_\bullet) \rightarrow CPH_\bullet(\tilde{E}_\bullet), \quad L^{[p]} : \overline{CPH}^f_\bullet(\tilde{E}_\bullet) \rightarrow \overline{CPH}^{[p]}_\bullet(\tilde{E}_\bullet),$$

such that for any complex  $E_\bullet$  in the category  $\text{Fun}(\Lambda_p, R)$ , we have  $J \circ R = R \circ J^f$  and  $\bar{J} \circ L^{[p]} = L^{[p]} \circ \bar{J}^f$ .

**Definition 5.14.** A complex  $E_\bullet$  in the category  $\text{Fun}(\Delta^\circ, R[\mathbb{Z}/p\mathbb{Z}])$  is *convergent* if the maps (5.24) are quasiisomorphisms.

**Lemma 5.15.** Assume given a complex  $E_\bullet$  in the category  $\text{Fun}(\Lambda_p, R)$  such that  $\tilde{j}_p^* E_\bullet$  is convergent in the sense of Definition 5.14. Then the maps

$$R : CP_\bullet^f(E_\bullet) \rightarrow CP_\bullet(E_\bullet), \quad L^{[p]} : \overline{CP}^f(E_\bullet) \rightarrow \overline{CP}^{[p]}(E_\bullet)$$

are quasiisomorphisms.

*Proof.* By Lemma 4.7 and its completed versions, it suffices to prove that the maps  $J, \bar{J}, J^f, \bar{J}^f$  are quasiisomorphisms: then the maps  $R, L^{[p]}$  are retracts of the maps (5.24), and we are done. For the maps  $J, J^f$  and  $\bar{J}^f$  this is clear, since they are all versions of the quasiisomorphism (3.10). The case of the map  $\bar{J}$  is more delicate, since the definition of  $CPH_\bullet^{[p]}(E_\bullet)$  and  $CPH_\bullet^{[p]}(\tilde{j}_p^* E_\bullet)$  involves completions, and the map (3.10) is not a filtered quasiisomorphism with respect to the standard filtrations. However, we can also filter  $cph_\bullet(E_\bullet) = CH_\bullet(\pi_{pb} E_\bullet \otimes \mathbb{K}_\bullet)$  by setting

$$V^n cph_\bullet(E_\bullet) = cc_\bullet(V^n E_\bullet^\flat \otimes \mathbb{K}_\bullet),$$

where the filtration  $V^n$  on  $E_\bullet^\flat = \pi_{pb} E_\bullet$  is as in Subsection 5.2. Then  $V^\bullet$  is commensurable with the standard filtration in every cohomological degree, hence gives the same completion  $\overline{CP}^{[p]}_\bullet(E_\bullet)$ , and the induced filtration on  $cph_\bullet(\tilde{j}_p^* E_\bullet)$  has completion  $\overline{CPH}^{[p]}_\bullet(\tilde{j}_p^* E_\bullet)$ . The map  $\bar{J}$  is now another instance of the quasiisomorphism (3.10).  $\square$

**5.4 Convergent complexes.** By virtue of Lemma 5.15, studying the comparison maps  $R, L^{[p]}$  reduces to studying convergent complexes in the category  $\text{Fun}(\Delta^\circ, R[\mathbb{Z}/p\mathbb{Z}])$ . We will need two results in this direction.

For any bicomplex  $E_{\bullet,\bullet}$  of  $R[\mathbb{Z}/p\mathbb{Z}]$ -modules, consider the Tate complex  $\tilde{C}_\bullet(\mathbb{Z}/p\mathbb{Z}, E_{\bullet,\bullet})$  of (2.12), and equip it with filtrations  $F^\bullet, F_1^\bullet, F_2^\bullet$  by

$$\begin{aligned} F^n \tilde{C}_\bullet(\mathbb{Z}/p\mathbb{Z}, E_{\bullet,\bullet}) &= \bigoplus_{i \geq n} \tilde{C}_\bullet(\mathbb{Z}/p\mathbb{Z}, E_{i,\bullet}) \subset \tilde{C}_\bullet(\mathbb{Z}/p\mathbb{Z}, E_{\bullet,\bullet}) \\ (5.25) \quad F_1^n \tilde{C}_\bullet(\mathbb{Z}/p\mathbb{Z}, E_{\bullet,\bullet}) &= \bigoplus_{i \geq n} \tilde{C}_i(\mathbb{Z}/p\mathbb{Z}, E_{\bullet,\bullet}) \subset \tilde{C}_\bullet(\mathbb{Z}/p\mathbb{Z}, E_{\bullet,\bullet}) \\ F_2^n \tilde{C}_\bullet(\mathbb{Z}/p\mathbb{Z}, E_{\bullet,\bullet}) &= \bigoplus_{i+pj \geq n} \tilde{C}_\bullet(\mathbb{Z}/p\mathbb{Z}, E_{i,j}) \subset \tilde{C}_\bullet(\mathbb{Z}/p\mathbb{Z}, E_{\bullet,\bullet}). \end{aligned}$$

Denote the completions of  $\tilde{C}_\bullet(\mathbb{Z}/p\mathbb{Z}, E_{\bullet,\bullet})$  with respect to filtrations  $F_1^\bullet, F_2^\bullet$  by  $C_\bullet^1(E_{\bullet,\bullet})$  and  $C_\bullet^2(E_{\bullet,\bullet})$ . Then  $F^\bullet$  induces a filtration on each of these two complexes, and we have natural maps

$$(5.26) \quad \lim_{\substack{\rightarrow \\ n}} F^{-n} C_\bullet^1(E_{\bullet,\bullet}) \rightarrow C_\bullet^1(E_{\bullet,\bullet}), \quad \lim_{\substack{\rightarrow \\ n}} F^{-n} C_\bullet^2(E_{\bullet,\bullet}) \rightarrow C_\bullet^2(E_{\bullet,\bullet}).$$

If  $E_{\bullet,\bullet} = CH_\bullet(E_\bullet)$  for a complex  $E_\bullet$  in the category  $\text{Fun}(\Delta^o, R[\mathbb{Z}/p\mathbb{Z}])$ , then we have  $\tilde{C}_\bullet(E_{\bullet,\bullet}) \cong cph_\bullet(E_\bullet)$ . We then have

$$C_\bullet^1(E_{\bullet,\bullet}) \cong CPH_\bullet(E_\bullet), \quad C_\bullet^2(E_{\bullet,\bullet}) \cong \overline{CPH}_\bullet^{[p]}(E_\bullet),$$

and the maps (5.26) are exactly the maps (5.24).

**Lemma 5.16.** *Assume given a complex  $E_\bullet$  in  $\text{Fun}(\Delta^o, R[\mathbb{Z}/p\mathbb{Z}])$ , and consider the complex  $CH_\bullet(E_\bullet)$  as a complex in  $C_\bullet(R[\mathbb{Z}/p\mathbb{Z}])$ , with terms  $E_\bullet([n])$ ,  $[n] \in \Delta$ . If  $CH_\bullet(E_\bullet)$  is strongly bounded from below in the sense of Definition 2.4, then  $E_\bullet$  is convergent in the sense of Definition 5.14.*

*Proof.* It suffices to prove that more generally, for any bicomplex  $E_{\bullet,\bullet}$  of  $R[\mathbb{Z}/p\mathbb{Z}]$ -modules that is strongly bounded from below with respect to the first index, the maps (5.26) are quasiisomorphisms. This is obvious: if  $E_{\bullet,\bullet}$  is contractible with respect to the first index, then all the complexes in (5.26) are contractible, and if  $E_{i,\bullet} = 0$  for  $i \gg 0$ , the limits in (5.26) stabilize at a finite step.  $\square$

Now note that for any bicomplex  $E_{\bullet,\bullet}$  of  $R[\mathbb{Z}/p\mathbb{Z}]$ -bimodules, the filtration  $F_1^\bullet$  of (5.25) induces a spectral sequence, and if  $E_{\bullet,\bullet} = CH_\bullet(E_\bullet)$  for some complex  $E_\bullet$  in  $\text{Fun}(\Delta^o, R[\mathbb{Z}/p\mathbb{Z}])$ , this spectral sequence reads as

$$(5.27) \quad CH_\bullet(E_\bullet)((u))\langle\varepsilon\rangle \Rightarrow CPH_\bullet(E_\bullet),$$

where  $\varepsilon$  is a formal generator of cohomological degree 1, as in (5.4). This is an analog of the standard spectral sequence of a cyclic object. To obtain an analog of the conjugate spectral sequence, we need a version of Definition 5.4.

**Definition 5.17.** A complex  $E_\bullet$  in  $\text{Fun}(\Delta^o, R[\mathbb{Z}/p\mathbb{Z}])$  is *tight* if  $E_\bullet([n])$  is a tight complex of  $R[\mathbb{Z}/p\mathbb{Z}]$ -modules for any  $[n] \in \Delta$ . For any tight complex  $E_\bullet$  in  $\text{Fun}(\Delta^o, R[\mathbb{Z}/p\mathbb{Z}])$ , the complex  $l(E_\bullet)$  is given by

$$l(E_\bullet)([n]) = l(E_\bullet([n]))$$

for any  $[n] \in \Delta$ .

Then for any tight complex  $E_\bullet$  in  $\text{Fun}(\Delta^o, R[\mathbb{Z}/p\mathbb{Z}])$ , we can consider the filtration on  $cph_\bullet(E_\bullet)$  induced by the filtration  $\tau^\bullet$  on  $\pi_{pb}E_\bullet^{[p]}$ . By Lemma 1.8, in every cohomological degree, this filtration is commensurable to the filtration  $F_2^\bullet$  of (5.25), so that we obtain a spectral sequence

$$(5.28) \quad CH_\bullet(l(E_\bullet))((u^{-1}))\langle \varepsilon \rangle \Rightarrow \overline{CPH}_\bullet^{[p]}(E_\bullet),$$

with the same meaning of  $\varepsilon$  as in (5.27).

**Lemma 5.18.** (i) *A finite extension of convergent complexes is convergent.*

(ii) *Assume given three tight complexes  $E_\bullet, E'_\bullet, E''_\bullet$  in  $\text{Fun}(\Delta^o, R[\mathbb{Z}/p\mathbb{Z}])$ , and two maps  $a : E'_\bullet \rightarrow E_\bullet, b : E_\bullet \rightarrow E''_\bullet$  such that both  $b \circ a$  and  $l(b \circ a)$  are quasiisomorphisms. Then if  $E_\bullet$  is convergent,  $E'_\bullet$  and  $E''_\bullet$  are convergent as well.*

*Proof.* (i) is clear. For (ii), note that by Lemma 2.11, the maps

$$\tilde{\pi}_{pb}E'_\bullet \rightarrow \tilde{\pi}_{pb}E''_\bullet, \quad \check{\pi}_{pb}E'_\bullet \rightarrow \check{\pi}_{pb}E''_\bullet$$

induced by  $b \circ a$  are quasiisomorphisms (for the second map, note that the stupid filtrations on  $E'_\bullet, E''_\bullet$  are commensurable to their  $p$ -th rescaling, and by Lemma 1.8, the induced filtrations on  $\pi_{pb}E'_\bullet, \pi_{pb}E''_\bullet$  are in turn commensurable to the filtration  $\tau^\bullet$  in every cohomological degree). Therefore by (5.20), the corresponding maps

$$CPH_\bullet^f(E'_\bullet) \rightarrow CPH_\bullet^f(E''_\bullet), \quad \overline{CPH}_\bullet^f(E'_\bullet) \rightarrow \overline{CPH}_\bullet^f(E''_\bullet)$$

are also quasiisomorphisms. Moreover, we have convergent spectral sequences (5.27), (5.28), so that the maps

$$CPH_\bullet(E'_\bullet) \rightarrow CPH_\bullet(E''_\bullet), \quad \overline{CPH}_\bullet^{[p]}(E'_\bullet) \rightarrow \overline{CPH}_\bullet^{[p]}(E''_\bullet)$$

induced by  $b \circ a$  are also quasiisomorphisms. Thus the maps (5.24) for  $E'_\bullet$  and  $E''_\bullet$  are retracts of the corresponding maps for  $E_\bullet$ , and we are done.  $\square$

**5.5 Characteristic 2.** In Subsection 4.2 and throughout Section 5, we have assumed that our base ring  $R$  is annihilated by an odd prime  $p$ . Let us now describe what happens if  $p = 2$ .

Note right away that since we are in characteristic 2, signs do not matter. In particular, there is no difference between the twisted complexes (2.14) and their untwisted versions.

The first problem occurs in the proof of Proposition 4.4, and specifically, in (4.8). If  $p = 2$ , then  $(p - 1)n = n$  is no longer necessarily divisible by 2, and it can happen that  $\mathrm{gr}_F^{i-np} \nu_p$  is a map from the Tate complex to its odd homological shift. However, this only happens if  $n$  is odd. For any  $n \geq 1$  and any  $R[\mathbb{Z}/2n\mathbb{Z}]$ -module  $E$ , we have a functorial map of complexes

$$(5.29) \quad \varepsilon_{\bullet} : C_{\bullet}(\mathbb{Z}/2n\mathbb{Z}, E) \rightarrow C_{\bullet+1}(\mathbb{Z}/2n\mathbb{Z}, E)$$

given by

$$\varepsilon_i = \begin{cases} \mathrm{id} + \sigma^2 + \cdots + \sigma^{2(n-1)}, & i = 2j, \\ \mathrm{id}, & i = 2j + 1, \end{cases}$$

where  $\sigma \in \mathbb{Z}/2n\mathbb{Z}$  is the generator. If  $n$  is odd, the map  $\varepsilon_{\bullet}$  is a quasiisomorphism. Therefore for odd  $n$ , we redefine the constants  $a_n, b_n$  by setting

$$\mathrm{gr}_F^{i-np} \nu_p \circ \varepsilon_{\bullet} = f(a_n, b_n)$$

instead of (4.8), and again, it suffices to prove that all the maps  $f(a_n, b_n)$  are quasiisomorphisms. Since in the case  $E = k$  we obviously have  $\varepsilon_{\bullet} = \mathrm{id}$  for all odd  $n$ , the rest of the proof of Proposition 4.4 goes through without any changes. The other results in Subsection 4.2 also go through with exactly the same proofs.

The real problem occurs in Section 5 and affects Subsection 5.1 and Subsection 5.2 (everything in Subsection 5.3 and Subsection 5.4 works for any prime). If  $p = 2$ , then the cohomology  $H^{\bullet}(\mathbb{Z}/p\mathbb{Z}, k)$  is still given by (5.4), but the multiplication is different: instead of  $\varepsilon^2 = 0$ , we have  $\varepsilon^2 = u$ , the periodicity generator. In fact, the action of the generator  $\varepsilon$  on the Tate complex is explicitly given by the quasiisomorphism (5.29). Thus in particular,  $\varepsilon_{\mathrm{odd}}$  is always an isomorphism, so that every  $R[\mathbb{Z}/2\mathbb{Z}]$ -module  $E$  is automatically tight. But on the downside,  $\varepsilon_{\mathrm{even}}$  is never equal to 0 – conversely, it is also an invertible map. Thus Lemma 5.7, Lemma 5.9 and Proposition 5.6 completely break down.

To alleviate the situation, let us prove a weaker version of Lemma 5.9 that works for all primes.

Assume that the base ring  $R$  is annihilated by a prime  $p$ , and assume given a complex  $E_{\bullet}$  in  $\mathrm{Fun}(\Lambda_p, R)$ . Assume further that  $E_{\bullet}$  is tight in the sense of Definition 5.2 (if  $p = 2$ , this just means that for any  $[n] \in \Lambda_p$  and any odd integer  $i$ ,  $E_i([n])$  is a free  $R[\mathbb{Z}/p\mathbb{Z}]$ -module). As in Subsection 5.2,



denote  $E_{\bullet}^{\flat} = \pi_{pb} E_{\bullet}^{[p]}$ ,  $\tilde{E}_{\bullet} = E_{\bullet} \otimes_k \tilde{\mathbb{K}}_{\bullet}$ ,  $\tilde{E}_{\bullet}^{\flat} = \pi_{pb} \tilde{E}_{\bullet}$ , and filter  $CC_{\bullet}(E_{\bullet}^{\flat})$  and  $CC_{\bullet}(\tilde{E}_{\bullet}^{\flat})$  by setting

$$(5.30) \quad V^{\bullet} CC_{\bullet}(E_{\bullet}^{\flat}) = CC_{\bullet}(V^{\bullet} E_{\bullet}^{\flat}), \quad V^{\bullet} CC_{\bullet}(\tilde{E}_{\bullet}^{\flat}) = CC_{\bullet}(V^{\bullet} \tilde{E}_{\bullet}^{\flat}),$$

where the filtration  $V^{\bullet}$  in the right-hand side is the filtration (5.12). Denote by  $\widehat{CC}_{\bullet}(E_{\bullet}^{\flat})$  the completion of the complex  $CC_{\bullet}(E_{\bullet}^{\flat})$  with respect to the filtration (5.30), and analogously for  $\tilde{E}_{\bullet}$ . Then as in Subsection 5.2, the map (3.12) induces a filtered map

$$\langle CC_{\bullet}(E_{\bullet}^{\flat}), V^{\bullet} \rangle \rightarrow \langle cp_{\bullet}(E_{\bullet}), V^{\bullet} \rangle,$$

where  $V^{\bullet}$  in the right-hand side is the conjugate filtration of Definition 5.1. Passing to the completions, we obtain a functorial map

$$(5.31) \quad \widehat{CC}_{\bullet}(E_{\bullet}^{\flat}) \rightarrow \overline{CP}_{\bullet}^{[p]}(E_{\bullet}).$$

**Lemma 5.19.** *The map (5.31) is a quasiisomorphism.*

*Proof.* If we equip the complex  $E_{\bullet}^{\flat} \otimes \mathbb{K}_{\bullet}$  with the shift  $V_{-1}^{\bullet} E_{\bullet}^{\flat} \otimes \mathbb{K}_{\bullet}$  of the filtration  $V^{\bullet}$ , then the map  $\delta^{\flat}$  of (5.13) is tautologically a filtered map, and the composition  $\gamma^{\flat} \circ \delta^{\flat}$  factors as

$$\langle \tilde{E}_{\bullet}^{\flat}, V^{\bullet} \rangle \xrightarrow{\iota} \langle E_{\bullet}^{\flat}, V^{\bullet} \rangle \xrightarrow{\text{id}} \langle E_{\bullet}^{\flat}, V_{-1}^{\bullet} \rangle,$$

where  $\iota$  is a filtered quasiisomorphism, and  $\text{id}$  is the identity map considered as a filtered map from  $V^{\bullet}$  to  $V_{-1}^{\bullet}$ . Since all shifts of a given filtration are commensurable, the map

$$\gamma \circ \delta : \widehat{CC}_{\bullet}(\tilde{E}_{\bullet}^{\flat}) \rightarrow \widehat{CC}_{\bullet}(E_{\bullet}^{\flat})$$

induced by the composition  $\gamma^{\flat} \circ \delta^{\flat}$  is a quasiisomorphism. Then as in the proof of Lemma 5.9, the map

$$\widehat{CC}_{\bullet}(E_{\bullet}^{\flat} \otimes \mathbb{K}_{\bullet}) \rightarrow \overline{CPH}_{\bullet}^{[p]}(E_{\bullet})$$

is a quasiisomorphism by Lemma 3.11, and the map (5.31) is a retract of this map.  $\square$

As a corollary of Lemma 5.19, we obtain a functorial spectral sequence

$$(5.32) \quad HC_{\bullet}(\text{gr}_V^{\bullet} E_{\bullet}^{\flat}) \Rightarrow \overline{HP}_{\bullet}^{[p]}(E_{\bullet})$$

induced by the filtration (5.30). Unfortunately, if  $p = 2$ , the identification of its first term presents a problem. Namely, let  $a_{\text{odd}}, a_{\text{even}} \in \text{Ext}^2(\mathbf{l}(E_\bullet), \mathbf{l}(E_\bullet))$  be the extension classes given by successive associated graded quotients of the filtration  $\tau^\bullet$  on 2-the periodic complex  $E_\bullet^\flat$ , so that  $\text{gr}_V^0 E_\bullet^\flat$  is an extension of  $\mathbf{l}(E_\bullet)$  by  $\mathbf{l}(E_\bullet)$  given by the class  $a_{\text{even}}$ . If  $p$  is odd, then Lemma 5.7 shows that  $a_{\text{even}} = u \cdot \text{id}$ , where  $u \in H^2(\Lambda, k)$  is the periodicity class. This implies that  $\text{gr}_V^0 E_\bullet^\flat$  is isomorphic to  $\mathbf{l}(E_\bullet) \otimes \mathbb{K}_\bullet$  in the derived category  $\mathcal{D}(\Lambda, R)$ , so that the left-hand side of (5.32) is identified with  $HH_\bullet(\mathbf{l}(E_\bullet))$ . If  $p = 2$ , then it is easy to modify the arguments of Lemma 5.7 to show that

$$a_{\text{even}} + a_{\text{odd}} = u \cdot \text{id}.$$

Ideally, we would have  $a_{\text{odd}} = 0$ , so that  $a_{\text{even}} = u \cdot \text{id}$ , as in the case of an odd prime. However, it certainly cannot be true for any tight complex  $E_\bullet$ . Indeed, we can just take a single object  $E \in \text{Fun}(\Lambda_p, R)$ , so that the tightness condition becomes trivial, and consider a short exact sequence

$$0 \longrightarrow E' \longrightarrow P \longrightarrow E \longrightarrow 0$$

with some projective object  $P \in \text{Fun}(\Lambda_p, R)$ . Then  $\mathbf{l}(P) = 0$  and  $\mathbf{l}(E) = \mathbf{l}(E')[1]$ , with the roles of the classes  $a_{\text{odd}}$  and  $a_{\text{even}}$  for  $E'$  played by  $a_{\text{even}}$  resp.  $a_{\text{odd}}$ . Thus we cannot have  $a_{\text{odd}} = 0$  for  $E$  and for  $E'$  at the same time.

We expect that for complexes  $E_\bullet$  that come from DG algebras and DG categories, we do have  $a_{\text{odd}} = 0$ , so that the spectral sequence (5.32) has the form prescribed by Proposition 5.6. However, we have not been able to prove it. Therefore we restrict ourselves to the following observation. Since the filtration  $\tau^\bullet$  on  $E_\bullet^\flat$  is commensurable to its rescaling  $V^\bullet$ , we can consider a spectral sequence induced by  $\tau^\bullet$  instead of  $V^\bullet$ . Thus for any prime  $p$  and tight complex  $E_\bullet$  in  $\text{Fun}(\Lambda_p, R)$ , the isomorphism (5.31) trivially yields a spectral sequence

$$(5.33) \quad HC_\bullet(\mathbf{l}(E_\bullet))((u^{-1}))\langle \varepsilon \rangle \Rightarrow \overline{HP}_\bullet^{[p]}(E_\bullet)$$

where  $u, \varepsilon$  are formal variables of cohomological degrees 2 and 1, as in (5.27).

## 6 DG categories.

**6.1 The setup.** We recall some basic facts about DG algebras and DG categories (the standard reference is [Ke]).

By a *DG algebra*  $A_\bullet$  over a commutative ring  $R$  we will understand an associative unital differential-graded algebra over  $R$  considered up to a

quasiisomorphism. In particular, we will tacitly assume that all DG algebras are  $h$ -projective as complexes of modules over  $R$ , and all individual terms  $A_i$ ,  $i \in \mathbb{Z}$  are flat  $R$ -modules (this can be achieved, for example, by choosing a cofibrant representative with respect to the standard model structure).

For any DG algebra  $A_\bullet$ , we denote by  $\mathcal{D}(A_\bullet)$  the derived category of left DG modules over  $A_\bullet$ . Any DG algebra map  $f : A_\bullet \rightarrow A'_\bullet$  induces by pullback a natural triangulated functor  $f_* : \mathcal{D}(A'_\bullet) \rightarrow \mathcal{D}(A_\bullet)$ , and it has a left-adjoint  $f^* : \mathcal{D}(A_\bullet) \rightarrow \mathcal{D}(A'_\bullet)$  sending  $M_\bullet$  to  $A'_\bullet \otimes_{A_\bullet} M_\bullet$ . The map  $f$  is a *derived Morita equivalence* if  $f^*$  and  $f_*$  are an adjoint pair of equivalences of categories. Every quasiisomorphism is a derived Morita equivalence.

A *DG category*  $A_\bullet$  over  $R$  is “an algebra with several objects” — we have a collection  $S$  of objects, and a collection of Hom-complexes  $A_\bullet(s, s')$  for any two objects  $s, s'$ , equipped with the associative composition maps and identity elements  $\text{id}_s \in A_0(s, s)$  for any object  $s$ . If  $S$  has exactly one element, then  $A_\bullet$  is simply a DG algebra. A DG category is small if  $S$  is a set. As in the DG algebra case, we consider small DG categories up to a quasiequivalence, and we will tacitly assume that the Hom-complexes  $A_\bullet(-, -)$  in a small DG category  $A_\bullet$  are  $h$ -projective complexes of flat  $R$ -modules. A *module*  $M_\bullet$  over a DG category  $A_\bullet$  is a contravariant DG functor from  $A_\bullet$  to the DG category of complexes of  $R$ -modules; explicitly, it is given by a collection of complexes  $M_\bullet(s)$ ,  $s \in S$ , and structure maps  $A_\bullet(s', s) \otimes_R M_\bullet(s) \rightarrow M_\bullet(s')$ . DG modules over a small DG category  $A_\bullet$  form a triangulated derived category  $\mathcal{D}(A_\bullet)$ , and for any DG functor  $f : A_\bullet \rightarrow A'_\bullet$  between small DG categories, we have an adjoint pair of natural triangulated functors  $f^* : \mathcal{D}(A_\bullet) \rightarrow \mathcal{D}(A'_\bullet)$ ,  $f_* : \mathcal{D}(A'_\bullet) \rightarrow \mathcal{D}(A_\bullet)$ . The functor  $f$  is a *derived Morita equivalence* if  $f^*$ ,  $f_*$  are adjoint equivalences of categories. More generally, a sequence

$$(6.1) \quad A'_\bullet \xrightarrow{f} A_\bullet \xrightarrow{g} A''_\bullet$$

of DG categories and DG functors is a *localization sequence* if  $g^* \circ f^* = 0$ ,  $f^*$  and  $g_*$  are fully faithful, and  $\langle f^*(\mathcal{D}(A'_\bullet)), g_*(\mathcal{D}(A''_\bullet)) \rangle$  is a semiorthogonal decomposition of the triangulated category  $\mathcal{D}(A_\bullet)$ . For example, one can take a small DG category  $A_\bullet$  and a set  $S$  of objects in  $A_\bullet$ , let  $A'_\bullet$  be the full subcategory  $A_\bullet(S) \subset A_\bullet$  spanned by  $S$ , and let  $A''_\bullet = A_\bullet/A'_\bullet$  be the *Drinfeld quotient* [Dr] obtained by formally adding to  $A_\bullet$  morphisms  $h_s : s \rightarrow s$ ,  $s \in S$  of homological degree 1 such that  $d(h_s) = \text{id}_s$ . Then the sequence

$$(6.2) \quad A'_\bullet \longrightarrow A_\bullet \longrightarrow A_\bullet/A'_\bullet$$

is a localization sequence. This example is universal: every localization sequence (6.1) is derived-Morita equivalent to a sequence (6.2).

A DG module  $M_\bullet$  over a DG category  $A_\bullet$  is *perfect* if it is compact as an object in  $\mathcal{D}(A_\bullet)$  (that is,  $\text{Hom}(M_\bullet, -)$  commutes with arbitrary sums). For any object  $s \in S$  of the category  $A_\bullet$ , we have the representable module  $A_\bullet^s$  given by

$$A_\bullet^s(s') = A_\bullet(s', s), \quad s, s' \in S.$$

Representable modules are perfect, and any perfect module is a retract of a finite extension of shifts of representable DG modules (“retract” here is understood in the derived category sense, that is,  $M_\bullet$  is a retract of  $M'_\bullet$  if we have a module  $M''_\bullet$  and maps  $a : M''_\bullet \rightarrow M'_\bullet$ ,  $b : M'_\bullet \rightarrow M_\bullet$  such that  $b \circ a$  is a quasiisomorphism). A small DG category  $A_\bullet$  is *compactly generated* if there exists a perfect DG module  $M_\bullet$  that weakly generates  $\mathcal{D}(A_\bullet)$  – that is, for any  $N_\bullet \in \mathcal{D}(A_\bullet)$ ,  $\text{Hom}(M_\bullet, N_\bullet) = 0$  implies  $N_\bullet = 0$ .

For any set  $S$  of objects in a DG category  $A_\bullet$ , we denote by  $A_\bullet^S$  the complex given by

$$(6.3) \quad A_\bullet^S = \bigoplus_{s, s' \in S} A_\bullet(s, s').$$

If  $S$  is finite, then this is a unital associative DG algebra over  $R$ , and we have a natural equivalence of derived categories

$$\mathcal{D}(A_\bullet^S) \cong \mathcal{D}(A_\bullet(S)),$$

where  $A_\bullet(S) \subset A_\bullet$  is the full DG subcategory spanned by  $S$ . The category  $A_\bullet$  is compactly generated if and only if the embedding  $A_\bullet(S) \rightarrow A_\bullet$  is a derived Morita equivalence for a finite set  $S$ ; in this case,  $\mathcal{D}(A_\bullet)$  is equivalent to the derived category  $\mathcal{D}(A_\bullet^S)$  of the DG algebra  $A_\bullet^S$ .

For any two small DG categories  $A_\bullet, A'_\bullet$  with sets of objects  $S, S'$ , the *tensor product*  $A_\bullet \otimes_R A'_\bullet$  is the DG category with set of objects  $S \times S'$  and Hom-complexes given by

$$(A_\bullet \otimes_R A'_\bullet)(s_1 \times s'_1, s_2 \times s'_2) = A_\bullet(s_1, s_2) \otimes_R A'_\bullet(s'_1, s'_2)$$

for any  $s_1, s_2 \in S, s'_1, s'_2 \in S'$ . A *DG bimodule*  $M_\bullet$  over  $A_\bullet$  is a DG-module over the product  $A_\bullet^\circ \otimes_R A_\bullet$  of  $A_\bullet$  with its opposite DG category  $A_\bullet^\circ$ . An example of a DG bimodule is the diagonal bimodule  $A_\bullet$ . A DG bimodule  $M_\bullet$  is perfect if it is perfect as a DG module over  $A_\bullet^\circ \otimes_R A_\bullet$ , and a small DG category  $A_\bullet$  is *smooth* if the diagonal bimodule  $A_\bullet$  is perfect. Smoothness is a derived-Morita invariant property: for any derived Morita equivalence  $f : A_\bullet \rightarrow A'_\bullet$  of small DG categories,  $A_\bullet$  is smooth if and only if  $A'_\bullet$  is smooth. It is convenient to introduce another Morita-invariant property of

DG categories that does not seem to have a standard name in the literature (although it did appear, for example, in [O], where it was emphasized as an important feature of DG categories of geometric origin).

**Definition 6.1.** A DG category  $A_\bullet$  over a ring  $R$  is *bounded from above* resp. *bounded from below* if for any objects  $s, s' \in A_\bullet$  and any  $R$ -module  $M$ , the complex  $A_\bullet(s, s') \otimes_R M$  is bounded from above resp. below in the sense of Definition 2.9. A DG category is *bounded* if it is bounded both from above and from below.

Again, for any derived Morita equivalence  $f : A_\bullet \rightarrow A'_\bullet$  of small DG categories,  $A_\bullet$  is bounded from above resp. below if and only if  $A'_\bullet$  is bounded from above resp. below.

For any DG algebra  $A_\bullet$ , one defines a complex  $A_\bullet^\natural$  in  $\text{Fun}(\Lambda, R)$  by setting

$$A_\bullet^\natural([n]) = A_\bullet^{\otimes_{R^n}}, \quad [n] \in \Lambda,$$

where terms in the product are numbered by vertices  $v \in V([n])$ . For any map  $f : [n'] \rightarrow [n]$ , the corresponding map  $A_\bullet^\natural(f)$  is given by

$$(6.4) \quad A_\bullet^\natural(f) = \bigotimes_{v \in V([n])} m_{f^{-1}(v)},$$

where for every finite totally ordered set  $S$ , we let  $m_S : A_\bullet^{\otimes_{R^S}} \rightarrow A_\bullet$  be the multiplication map. If  $S$  is empty, we let  $A_\bullet^{\otimes_{R^S}} = R$ , and  $m_S : R \rightarrow A_\bullet$  is the embedding map of the unity element  $1 \in A_0$ .

To extend it to DG categories, for any small DG category  $A_\bullet$  with the set of objects  $S$ , one sets

$$A_\bullet^\natural([n]) = \bigoplus_{s_1, \dots, s_n \in S} A_\bullet(s_1, s_2) \otimes_R \cdots \otimes_R A_\bullet(s_{n-1}, s_n) \otimes_R A_\bullet(s_n, s_1)$$

for any object  $[n] \in \Lambda$ . Then if  $S$  is finite,  $A_\bullet^\natural([n])$  is canonically a direct summand of  $(A_\bullet^S)^\natural([n])$ , where  $A_\bullet^S$  is the DG algebra (6.3), and the structure maps (6.4) induce maps between  $A_\bullet^\natural([n])$  turning  $A_\bullet^\natural$  into a complex in  $\text{Fun}(\Lambda, R)$ . In the general case, we let

$$A_\bullet^\natural = \lim_{\substack{\longrightarrow \\ S}} A_\bullet(S)^\natural,$$

where the limit is over all finite sets  $S$  of objects in  $A_\bullet$ .

The *Hochschild homology*  $HH_*(A_*)$ , the *cyclic homology*  $HC_*(A_*)$ , and the *periodic cyclic homology*  $HP_*(A_*)$  of a small DG category  $A_*$  is defined by means of the complex  $A_*^\natural$ : we denote

$$CH_*(A_*) = CH_*(A_*^\natural), \quad CC_*(A_*) = CC_*(A_*^\natural), \quad CP_*(A_*) = CP_*(A_*^\natural),$$

and we let  $HH_*(A_*)$ ,  $HC_*(A_*)$ ,  $HP_*(A_*)$  be the homology groups of these complexes. Any DG functor  $f : A_* \rightarrow A'_*$  between small DG categories induces maps  $HH_*(A_*) \rightarrow HH_*(A'_*)$ ,  $HC_*(A_*) \rightarrow HC_*(A'_*)$ ,  $HP_*(A_*) \rightarrow HP_*(A'_*)$ . If  $f$  is a derived Morita equivalence, then all three maps are isomorphisms. More generally, a localization sequence (6.1) induces a distinguished triangle

$$CH_*(A'_*) \longrightarrow CH_*(A_*) \longrightarrow CH_*(A''_*) \longrightarrow$$

of Hochschild homology complexes and the induced long exact sequence of Hochschild homology groups, and we have analogous distinguished triangles and long exact sequences for cyclic and for periodic cyclic homology. One shortens this by saying that  $HH_*(-)$ ,  $HC_*(-)$ , and  $HP_*(-)$  are *additive invariants* of small DG categories.

**6.2 Statements.** Assume given a small DG category  $A_*$  over a commutative ring  $R$ , and consider the corresponding complex  $A_*^\natural$  in  $\text{Fun}(A, R)$ .

**Definition 6.2.** The *co-periodic cyclic homology*  $\overline{HP}_*(A_*)$  is given by

$$\overline{HP}_*(A_*) = \overline{HP}_*(A_*^\natural),$$

where the right-hand side is as in Definition 3.2, and the *co-periodic cyclic complex*  $\overline{CP}_*(A_*)$  is the complex  $\overline{CP}_*(A_*^\natural)$ . The *polynomial periodic cyclic homology*  $hp_*(A_*)$  is the homology of the complex  $cp_*(A_*^\natural)$  of Definition 3.3, and and the *restricted periodic and co-periodic cyclic homology*  $HP_*^f(A_*)$ ,  $\overline{HP}_*^f(A_*)$  are the homology of the complexes  $CP^f(A_*^\natural)$ ,  $\overline{CP}^f(A_*^\natural)$  of Definition 3.5.

**Lemma 6.3.** *For any small DG category  $A_*$  over a commutative ring  $R$ , we have*

$$hp_*(A_*) \otimes \mathbb{Q} = HP_*^f(A_*) \otimes \mathbb{Q} = \overline{HP}_*^f(A_*) \otimes \mathbb{Q} = \overline{HP}_*(A_*) \otimes \mathbb{Q} = 0.$$

*Proof.* Lemma 3.7 and Corollary 3.9.  $\square$

Now we formulate our main technical result about co-periodic cyclic homology. Assume that the base ring  $R$  is annihilated by a prime  $p$ . For any  $R$ -module  $M$ , denote by  $M^{(1)}$  its Frobenius twist — that is, the module  $V \otimes_R R^{(1)}$ , where  $R^{(1)}$  is  $R$  considered as an algebra over itself via the absolute Frobenius map  $R \rightarrow R$ ,  $r \mapsto r^p$ . For any DG category  $A_\bullet$  over  $R$ , denote by  $HH_\bullet^{(1)}(A_\bullet)$  resp.  $HC_\bullet^{(1)}(A_\bullet)$  the homology groups of the complexes  $CH_\bullet(A_\bullet)^{(1)}$  resp.  $CC_\bullet(A_\bullet)^{(1)}$ .

**Proposition 6.4.** *Assume given a small DG category  $A_\bullet$  over a ring  $R$  annihilated by a prime  $p$ . Then we have a convergent spectral sequence*

$$(6.5) \quad HC_\bullet^{(1)}(A_\bullet)((u^{-1}))\langle\varepsilon\rangle \Rightarrow \overline{HP}_\bullet(A_\bullet),$$

where  $u, \varepsilon$  are formal variables of cohomological degrees 2 and 1, as in (5.27). Moreover, if  $p \neq 2$ , we have a convergent spectral sequence

$$(6.6) \quad HH_\bullet^{(1)}(A_\bullet)((u^{-1})) \Rightarrow \overline{HP}_\bullet(A_\bullet).$$

Both spectral sequences are functorial in  $A_\bullet$ .

The spectral sequence (6.6) is called the *conjugate spectral sequence* for the DG category  $A_\bullet$ . We will prove Proposition 6.4 in Subsection 6.4. For now, we use it to prove the following result.

**Theorem 6.5.** *Assume that the commutative ring  $R$  is Noetherian. Then for any derived Morita equivalence  $f : A_\bullet \rightarrow A'_\bullet$  between small DG categories over  $R$ , the induced map*

$$\overline{HP}_\bullet(A_\bullet) \rightarrow \overline{HP}_\bullet(A'_\bullet)$$

*is an isomorphism.*

*Proof.* Since  $R$  is Noetherian, the derived category of  $R$ -modules is generated by residue fields of localizations of  $R$ . Therefore it suffices to prove the claim after taking product with such a residue field. In other words, we may assume right away that  $R = k$  is a field. Then if  $k$  contains  $\mathbb{Q}$ , we are done by Lemma 6.3, and if not, we can apply the spectral sequence (6.5) and the corresponding property of the cyclic homology functor  $HC_\bullet(-)$ .  $\square$

Moreover, we can also prove a stronger statement — not only is co-periodic cyclic homology derived-Morita invariant, but it also gives an additive invariant of small DG categories over a fixed Noetherian ring.

**Theorem 6.6.** *Assume that the commutative ring  $R$  is Noetherian. Then any localization sequence (6.1) of small DG categories over  $R$  induces a long exact sequence*

$$\overline{HP}_\bullet(A'_\bullet) \longrightarrow \overline{HP}_\bullet(A_\bullet) \longrightarrow \overline{HP}_\bullet(A''_\bullet) \longrightarrow$$

*of co-periodic cyclic homology.*

*Proof.* By Theorem 6.5, we may assume that the localization sequence in question is of the form (6.2). Then by definition, the composition  $g \circ f : A'_\bullet \rightarrow A''_\bullet = A_\bullet/A'_\bullet$  factors through natural projection  $q : A'_\bullet \rightarrow A'_\bullet/A'_\bullet$  to the Drinfeld quotient  $A'_\bullet/A'_\bullet$ . Therefore if we denote by  $\overline{CP}_\bullet(A''_\bullet)$  the cone of the natural map of complexes

$$\overline{CP}_\bullet(A'_\bullet) \xrightarrow{f-q} \overline{CP}_\bullet(A_\bullet) \oplus \overline{CP}_\bullet(A'_\bullet/A'_\bullet),$$

then  $g$  induces a natural map of complexes

$$(6.7) \quad \overline{CP}'_\bullet(A''_\bullet) \rightarrow \overline{CP}_\bullet(A''_\bullet),$$

and since  $\overline{CP}_\bullet(A'_\bullet/A'_\bullet)$  is acyclic by Theorem 6.5, it suffices to prove that (6.7) is a quasiisomorphism. As in the proof of Theorem 6.5, it suffices to prove this when  $R = k$  is a field, and the statement then immediately follows from Lemma 6.3 if  $k$  contains  $\mathbb{Q}$  and from (6.5) otherwise.  $\square$

Finally, we also have a comparison result about different versions of periodic cyclic homology (the proof is in Subsection 6.4). Note that by definition, taking  $E_\bullet = A_\bullet^\natural$  in (3.23) induces a corresponding commutative diagram for any small DG category  $A_\bullet$ .

**Theorem 6.7.** *Assume given a DG algebra  $A_\bullet$  over a Noetherian commutative ring  $R$ .*

- (i) *If  $A_\bullet$  is bounded from above, then the map  $r : hp_\bullet(A_\bullet) \rightarrow \overline{HP}_\bullet^f(A_\bullet)$  of (3.23) is an isomorphism.*
- (ii) *If  $A_\bullet$  is bounded from below, then the map  $l : hp_\bullet(A_\bullet) \rightarrow HP_\bullet^f(A_\bullet)$  of (3.23) is an isomorphism.*
- (iii) *Assume that  $A_i = 0$  unless  $i \geq 0$ . Then the map  $R : \overline{HP}_\bullet^f(A_\bullet) \rightarrow \overline{HP}_\bullet(A_\bullet)$  of (3.23) is an isomorphism.*



- (iv) Assume that  $A_\bullet$  is smooth. Then the map  $R : \overline{HP}_\bullet^f(A_\bullet) \rightarrow \overline{HP}_\bullet(A_\bullet)$  of (3.23) is an isomorphism, and the map  $L : HP_\bullet^f(A_\bullet) \rightarrow HP_\bullet(A_\bullet)$  fits into a long exact sequence

$$HP_\bullet^f(A_\bullet) \xrightarrow{L} HP_\bullet(A_\bullet) \longrightarrow HP_\bullet(A_\bullet) \otimes \mathbb{Q} \longrightarrow .$$

Here DG algebra is treated as a DG category with one object, and “bounded from above/below” is understood in the sense of Definition 6.1. The condition in (iii) can also be replaced with its cohomological version (we can always replace  $A_\bullet$  with its truncation  $\tau^0 A_\bullet$ ). Either way, the condition is pretty strong. However, it does hold in some interesting cases, for example when  $A_\bullet$  is a unital associative algebra  $A$  placed in homological degree 0. In this case, Theorem 6.7 (i),(ii),(iii) actually gives

$$HP_\bullet^f(A) \cong hp_\bullet(A) \cong \overline{HP}_\bullet^f(A) \cong \overline{HP}_\bullet(A).$$

However, these groups are different from  $HP_\bullet(A)$  unless  $A$  has finite homological dimension.

**Corollary 6.8.** *Assume given a small compactly generated DG category  $A_\bullet$  over a Noetherian commutative ring  $R$ .*

- (i) *If  $A_\bullet$  is bounded from above and smooth, then the natural map  $R \circ r : hp_\bullet(A_\bullet) \rightarrow \overline{HP}_\bullet(A_\bullet)$  is an isomorphism.*
- (ii) *If  $A_\bullet$  is bounded from below and smooth, then the natural map  $L \circ l$  fits into a long exact sequence*

$$hp_\bullet(A_\bullet) \xrightarrow{L \circ l} HP_\bullet(A_\bullet) \longrightarrow HP_\bullet(A_\bullet) \otimes \mathbb{Q} \longrightarrow .$$

*In particular, if  $A_\bullet$  is bounded and smooth, and  $R \otimes \mathbb{Q} = 0$ , we have a natural isomorphism  $HP_\bullet(A_\bullet) \cong \overline{HP}_\bullet(A_\bullet)$ .*

*Proof.* If the set  $S$  of objects in  $A_\bullet$  is finite, all claims for the category  $A_\bullet(S)$  immediately follow from the Theorem 6.7 applied to the DG algebra  $A_\bullet^S$ . In the general case, finite subsets  $S_0 \subset S$  such that  $A_\bullet(S_0) \subset A_\bullet$  is a derived Morita equivalence are cofinal in the partially ordered set of all finite subsets in  $S$ . For every such subset  $S_0$ , we know the claims for  $A_\bullet(S_0)$ ; to finish the proof, notice that  $hp_\bullet(-)$  obviously commutes with filtered direct limits, while  $HP_\bullet(-)$  and  $\overline{HP}_\bullet(-)$  are derived-Morita invariant.  $\square$

**6.3 Tensor powers.** Before we prove Theorem 6.7 and Proposition 6.4, we need to make a digression about tensor power functors.

Fix a prime  $p$  and a commutative ring  $R$  such that  $pR = 0$ . For any flat  $R$ -module  $M$ , consider the  $p$ -fold tensor power  $M^{\otimes_{Rp}}$ , and let the cyclic group  $\mathbb{Z}/p\mathbb{Z}$  act on it by permutations, with the generator  $\sigma$  corresponding to the order- $p$  permutation  $\sigma : M^{\otimes_{Rp}} \rightarrow M^{\otimes_{Rp}}$ . We then have a natural trace map

$$M^{\otimes_{Rp}} \xrightarrow{\text{id} + \sigma + \dots + \sigma^{p-1}} M^{\otimes_{Rp}}.$$

Denote by  $C(M)$  the cokernel of this map, and consider the map

$$(6.8) \quad \psi : M^{(1)} \rightarrow C(M)$$

given by  $\psi(\lambda \otimes m) = \lambda m^{\otimes p}$ ,  $\lambda \in R$ ,  $m \in M$ .

**Lemma 6.9.** *The  $R[\mathbb{Z}/p\mathbb{Z}]$ -module  $M^{\otimes_{Rp}}$  is tight. The map  $\psi$  of (6.8) is an additive  $R$ -linear map, and it induces an isomorphism  $M^{(1)} \cong \mathsf{l}(M)$ .*

*Proof.* All the claims are obviously compatible with filtered colimits and retracts, so that we may assume that  $M = R[S]$  is the free  $R$ -module generated by a set  $S$ . The rest of the proof proceeds exactly as the special case when  $R$  is a field considered in [Ka3, Lemma 4.1]. Namely, decompose

$$(6.9) \quad M^{\otimes_{Rp}} = R[S^p] = M \oplus M',$$

where  $M' = R[S']$  is the free module spanned by complement  $S' = S^p \setminus S$  to the diagonal  $S \subset S^p$ . Then  $M'$  is a free  $R[\mathbb{Z}/p\mathbb{Z}]$ -module, and the  $\mathbb{Z}/p\mathbb{Z}$ -action on  $M$  is trivial, so that both are tight. Moreover, since  $\sigma(m^{\otimes p}) = m^{\otimes p}$  for any  $m \in M$ , the map  $\psi$  takes values in  $\mathsf{l}(M^{\otimes_{Rp}}) \subset C(M)$ . Moreover, its composition with the projection  $\mathsf{l}(M^{\otimes_{Rp}}) \rightarrow \mathsf{l}(M)$  onto the first summand in (6.9) is obviously an isomorphism. To finish the proof, it remains to notice that since  $M'$  is free over  $R[\mathbb{Z}/p\mathbb{Z}]$ , we have  $\mathsf{l}(M') = 0$ .  $\square$

Assume now given a complex  $M_\bullet$  of flat  $R$ -modules, and consider the  $p$ -fold tensor power  $M_\bullet^{\otimes_{Rp}}$ . Again, equip it with the permutation action of the group  $\mathbb{Z}/p\mathbb{Z}$ .

**Proposition 6.10.** *The complex  $M_\bullet^{\otimes_{Rp}}$  is a tight complex of  $R[\mathbb{Z}/p\mathbb{Z}]$ -modules in the sense of Definition 5.2, and we have a functorial isomorphism*

$$(6.10) \quad \mathsf{l}(M^{\otimes_{Rp}}) \cong M_\bullet^{(1)}.$$

*Proof.* As in Lemma 6.9, we may assume right away that  $M_i$  is a free  $R$ -module for any integer  $i$ . For any integer  $l$ , the degree- $l$  term of the complex  $M_{\bullet}^{\otimes_{RP}}$  decomposes as

$$(M_{\bullet}^{\otimes_{RP}})_l = \bigoplus_{i_1 + \dots + i_p = l} M_{i_1} \otimes_R \dots \otimes_R M_{i_p},$$

and the cyclic group  $\mathbb{Z}/p\mathbb{Z}$  permutes the indices  $i_1, \dots, i_p$ . If  $l$  is not divisible by  $p$ , then the  $\mathbb{Z}/p\mathbb{Z}$ -action on the set of indices such that  $i_1 + \dots + i_p = l$  is stabilizer-free, so that  $(M_{\bullet}^{\otimes_{RP}})_l$  is a free  $R[\mathbb{Z}/p\mathbb{Z}]$ -module. If  $l = np$  for some integer  $n$ , then we have

$$(6.11) \quad (M_{\bullet}^{\otimes_{RP}})_l = M_n^{\otimes_{RP}} \oplus M',$$

where  $M'$  is a free  $R[\mathbb{Z}/p\mathbb{Z}]$ -module. Since  $M_n^{\otimes_{RP}}$  is tight by Lemma 6.9,  $M_{\bullet}^{\otimes_{RP}}$  is therefore indeed a tight complex of  $R[\mathbb{Z}/p\mathbb{Z}]$ -modules. Moreover, (6.11) together with (5.8) give an identification

$$\mathsf{l}(M_{\bullet}^{\otimes_{RP}})_n \cong \mathsf{l}(M_n^{\otimes_{RP}}) \oplus \mathsf{l}(M') \cong \mathsf{l}(M_n^{\otimes_{RP}}),$$

and Lemma 6.9 then provides a functorial isomorphism

$$(6.12) \quad \tilde{\psi} : M_{\bullet}^{(1)} \cong \mathsf{l}(M_{\bullet}^{\otimes_{RP}})$$

of graded  $R$ -modules.

To check whether the map (6.12) commutes with the differential, consider first the special case when  $M_1 = M_0 = R$ ,  $M_i = 0$  otherwise, and  $d : M_1 \rightarrow M_0$  is the identity map. Then the complex  $M_{\bullet}^{\otimes_{RP}}$  is strongly acyclic as a complex of  $R[\mathbb{Z}/p\mathbb{Z}]$ -modules in the sense of Definition 2.8, so that  $\tilde{C}_{\bullet}(\mathbb{Z}/p\mathbb{Z}, M^{\otimes_{RP}})$  is an acyclic complex. Equip it with the filtration  $F^{\bullet}$  induced by the  $p$ -th rescaling of the stupid filtration on  $M_{\bullet}^{\otimes_{RP}}$ , and consider the spectral sequence induced by the filtration  $\tau^{\bullet}$  of (1.8). Then for dimension reasons, it degenerates at the term  $E_3$ , and the term  $E_2$  is the sum of shifts of the complex  $\mathsf{l}(M_{\bullet}^{\otimes_{RP}})$ . Thus the differential in this complex must be an isomorphism, and we have

$$(6.13) \quad \tilde{\psi} \circ d = ad \circ \tilde{\psi}$$

for some invertible element  $a \in R$ . But since the map (6.12) is functorial, the same equality must then hold for any complex  $M_{\bullet}$  of flat  $R$ -modules. Now to finish the proof, it suffices to define a graded isomorphism

$$(6.14) \quad \psi : M^{(1)} \cong \mathsf{l}(M_{\bullet}^{\otimes_{RP}})$$

by setting  $\psi = a^{-i}\tilde{\psi}$  in graded degree  $i$ , and observe that  $\psi$  commutes with the differentials by (6.13).  $\square$

**Remark 6.11.** The construction of the map (6.14) is obviously functorial in  $R$ , so that the constant  $a$  in (6.13) is actually an invertible element in the prime field  $k = \mathbb{Z}/p\mathbb{Z}$ . If  $p = 2$ , then of course  $a = 1$ , and if  $p$  is odd,  $a$  can be more explicitly described as follows: take  $M_\bullet$  with  $M_1 = M_0 = k$ ,  $M_i = 0$  otherwise,  $d = \text{id}$ , and note that its  $p$ -th tensor power represents by Yoneda a completely canonical class in the group  $\text{Exp}_{k[\mathbb{Z}/p\mathbb{Z}]}^{p-1}(k, k) = H^{p-1}(\mathbb{Z}/p\mathbb{Z}, k)$ . This class must be of the form  $au^{(p-1)/2}$ , where  $u \in H^2(\mathbb{Z}/p\mathbb{Z}, k)$  is the natural generator, and  $a \in k$  is some coefficient. This is our element  $a$ . Alternatively, the Steenrod  $p$ -th power of the generator  $\varepsilon \in H^1(S, k)$  of the first homology group of the circle  $S^1$  is equal to  $a\varepsilon$ . In view of the latter description, the value of  $a$  must be in the literature, but I could not find it.

Finally, we will need the following result about tensor powers of  $h$ -projective complexes.

**Lemma 6.12.** *Assume given a quasiisomorphism  $f : N_\bullet \rightarrow M_\bullet$  between  $h$ -projective complexes of flat  $R$ -modules. Then the  $p$ -th tensor power*

$$a^{\otimes p} : N_\bullet^{\otimes p} \rightarrow M_\bullet^{\otimes p}$$

*is a strong quasiisomorphism of complexes of  $R[\mathbb{Z}/p\mathbb{Z}]$ -modules in the sense of Definition 2.8.*

*Proof.* Since a quasiisomorphism of  $h$ -projective complexes is a chain-homotopy equivalence, it suffices to prove that for two chain-homotopic maps  $f_1, f_2 : N_\bullet \rightarrow M_\bullet$ , the tensor powers  $f_1^{\otimes p}, f_2^{\otimes p}$  give the same maps in the absolute derived category  $\mathcal{D}_{abs}(R[\mathbb{Z}/p\mathbb{Z}])$ . It obviously suffices to consider the universal situation: we let  $N_\bullet = M_\bullet \otimes I_\bullet$ , where  $I_\bullet$  is the complex with terms  $I_0 = \mathbb{Z} \oplus \mathbb{Z}$ ,  $I_{-1} = \mathbb{Z}$ ,  $I_i = 0$  otherwise,  $d : I_0 \rightarrow I_{-1}$  the difference map, and we let  $f_1, f_2$  be the maps induced by the projections  $I_0 \rightarrow \mathbb{Z}$  onto the two summands. Then we also have the map  $e : M_\bullet \rightarrow N_\bullet$  induced by the diagonal embedding  $\mathbb{Z} \rightarrow I_0$ , and  $f_1 \circ e = f_2 \circ e = \text{id}$ . Therefore  $f_1^{\otimes p} \circ e^{\otimes p} = f_2^{\otimes p} \circ e^{\otimes p} = \text{id}$ , and it suffices to prove that  $e^{\otimes p}$  is a strong quasiisomorphism. This is clear: the diagonal embedding  $\mathbb{Z} \rightarrow I_\bullet^{\otimes p}$  is a quasiisomorphism, and since the complex  $I_\bullet^{\otimes p}$  sits in a finite range of degrees, the quasiisomorphism must be strong.  $\square$

**Corollary 6.13.** *Assume that an  $h$ -projective complex  $M_\bullet$  of  $R$ -modules is bounded from above in the sense of Definition 2.9. Then its  $p$ -th tensor power  $M_\bullet^{\otimes_{RP}}$  is strongly bounded from above as a complex of  $R[\mathbb{Z}/p\mathbb{Z}]$ -modules.*

*Proof.* Choose  $n$  such that  $\tau^n M_\bullet \rightarrow M_\bullet$  is a quasiisomorphism, note that one can choose an  $h$ -projective replacement for  $\tau^n M_\bullet$  that is trivial in homological degrees  $< n$ , and apply Lemma 6.12.  $\square$

**6.4 Proofs.** We can now prove Proposition 6.4 and Theorem 6.7. In fact, Proposition 6.4 immediately follows from (5.33), Proposition 5.6 and the following result.

**Lemma 6.14.** *For any small DG category  $A_\bullet$  over a commutative ring  $R$  annihilated by a prime  $p$ , the complex  $A_\bullet^\natural$  in  $\text{Fun}(\Lambda, R)$  is  $p$ -adapted in the sense of Definition 5.4, and we have a natural identification*

$$(6.15) \quad \mathbf{l}(i_p^* A_\bullet^\natural) \cong (A_\bullet^\natural)^{(1)}.$$

*Proof.* If  $A_\bullet$  is a DG algebra — that is, the set  $S$  of its objects consists of one element — then for every  $[n] \in \Lambda_p$ , we have

$$A_\bullet^\natural(i_p([n])) \cong (A_\bullet^{\otimes_{R^n}})^{\otimes_{RP}}.$$

Proposition 6.10 immediately shows that  $A_\bullet^\natural$  is  $p$ -adapted, and the isomorphisms (6.10) provide the identification (6.15). If the set  $S$  is finite, then for any  $[n] \in \Lambda_p$ ,  $A_\bullet^\natural(i_p([n]))$  is a retract of  $A_\bullet^{S^\natural}(i_p([n]))$ . Since a retract of a tight complex is obviously tight,  $A_\bullet^\natural$  is again  $p$ -adapted, and the identification (6.15) for  $A_\bullet$  is induced by the corresponding identification for the DG algebra  $A_\bullet$ . In the general case, note that a filtered colimit of tight objects is obviously tight, and the functor  $\mathbf{l}(-)$  commutes with filtered colimits.  $\square$

The proof of Theorem 6.7 — specifically, of Theorem 6.7 (iv) — requires some preparations. Recall that the embedding  $j : \Delta^\circ \rightarrow \Lambda$  of (3.2) identifies  $\Delta^\circ$  with the category of objects  $[n] \in \Lambda$  equipped with a distinguished vertex  $v \in V([n])$ . For any bimodule  $M$  over an associative unital flat  $R$ -algebra  $A$ , we can define the simplicial  $R$ -modules  $(M/A)^\natural \in \text{Fun}(\Delta^\circ, R)$  by setting

$$(6.16) \quad (M/A)^\natural([n]) = A^{\otimes_{R^n-1}} \otimes_R M, \quad [n] \in \Delta^\circ$$

where the terms  $A$  in the product correspond to vertices  $v' \in V(j([n]))$  different from the distinguished vertex  $v$ , and  $M$  corresponds to  $v$ . The structure maps  $(M/A)^\natural(f)$ ,  $f : [n'] \rightarrow [n]$  are given by (6.4). The complex

$$CH_\bullet(A, M) = CH_\bullet((M/A)^\natural)$$

is then the standard Hochschild homology complex of the group algebra  $A$  with coefficients in  $M$ . In particular, if  $M = A^o \otimes_R A$  is a free  $A$ -bimodule, the complex  $CH_*(A, M)$  is chain-homotopy equivalent to  $HH_0(A, M) = A$  placed in degree 0. Moreover, for any integer  $l \geq 1$ , the  $l$ -fold tensor product  $A^{\otimes_{R^l}}$  is an associative algebra equipped with an action of the group  $\mathbb{Z}/l\mathbb{Z}$  generated by the longest permutation  $\sigma : A^{\otimes_{R^l}} \rightarrow A^{\otimes_{R^l}}$ . The  $l$ -fold tensor product  $M^{\otimes_{R^l}}$  is naturally a bimodule over  $A^{\otimes_{R^l}}$ . Denote by  $M_\sigma^{\otimes_{R^l}}$  the  $R$ -module  $M^{\otimes_{R^l}}$  considered as an  $A^{\otimes_{R^l}}$ -bimodule in the following way: the left multiplication is the standard one, and the right multiplication is twisted by  $\sigma$  — in other words, we have

$$a \cdot m \cdot a' = am\sigma(a'), \quad a, a' \in A^{\otimes_{R^l}}, m \in M^{\otimes_{R^l}}.$$

Then the object

$$(6.17) \quad (M/A)_l^\natural = (M_\sigma^{\otimes_{R^l}}/A^{\otimes_{R^l}})^\natural$$

carries a natural action of  $\mathbb{Z}/l\mathbb{Z}$  generated by the same permutation  $\sigma$  acting both on  $A^{\otimes_{R^l}}$  and on  $M^{\otimes_{R^l}}$ , so that it lies naturally in the category  $\text{Fun}(\Delta^o, R[\mathbb{Z}/l\mathbb{Z}])$ . We denote by

$$CH_*^l(A, M) = CH_*((M/A)_l^\natural)$$

the corresponding complex of  $R[\mathbb{Z}/l\mathbb{Z}]$ -modules, and we denote its homology by  $HH_*^l(A, M)$ . For example, if  $M = A^o \otimes_R A$ , we have

$$HH_0^l(A, M) = A^{\otimes_{R^l}},$$

and the complex  $CH_*^l(A, M)$  is chain-homotopy-equivalent to its degree-0 homology placed in degree 0.

For a bimodule  $M_\bullet$  over a DG algebra  $A_\bullet$ , we can apply (6.16) termwise and obtain the complex  $(M_\bullet/A_\bullet)^\natural$  in the category  $\text{Fun}(\Delta^o, R)$ . By definition, if  $A_\bullet$  is the diagonal bimodule, we have  $(A_\bullet/A_\bullet)^\natural \cong j^* A_\bullet^\natural$ . Moreover, for any integer  $l \geq 1$ , we can define the complex  $(M_\bullet/A_\bullet)_l^\natural$  in  $\text{Fun}(\Delta^o, R[\mathbb{Z}/l\mathbb{Z}])$  by (6.17), and we have

$$(6.18) \quad (A_\bullet/A_\bullet)_l^\natural \cong \tilde{j}_l^* i_l^* A_\bullet^\natural,$$

where  $\tilde{j}_l^*$  is the pullback functor (5.19).

**Lemma 6.15.** *Assume that the commutative ring  $R$  is annihilated by a prime  $p$ . Then for any DG algebra  $A_\bullet$  over  $R$  and any perfect  $A_\bullet$ -bimodule  $M_\bullet$  termwise flat over  $R$ , the complex  $(M_\bullet/A_\bullet)_p^\natural$  in  $\text{Fun}(\Delta^o, R[\mathbb{Z}/p\mathbb{Z}])$  is convergent in the sense of Definition 5.14.*

*Proof.* Say that an  $A_\bullet$ -bimodule is *finite free* if it is a finite sum of homological shifts of the bimodule  $A_\bullet^o \otimes_R A_\bullet$ , and say that it is *finite semifree* if it admits a filtration whose associated graded quotient is finite free. Since  $M_\bullet$  is perfect, we have a diagram

$$M''_\bullet \xrightarrow{a} M'_\bullet \xrightarrow{b} M_\bullet$$

of  $A_\bullet$ -bimodules such that  $M'_\bullet$  is finite semifree and  $b \circ a$  is a quasiisomorphism. We can also choose it in such a way that all the bimodules are complexes of flat  $R$ -modules. Then the composition  $b^{\otimes p} \circ a^{\otimes p}$  is also a quasiisomorphism. Moreover, by the same argument as in Lemma 6.14, Proposition 6.10 shows that for any  $A_\bullet$ -bimodule  $N_\bullet$  termwise flat over  $R$ , the complex  $(N_\bullet/A_\bullet)_p^\natural$  is tight, and we have

$$I((N_\bullet/A_\bullet)_p^\natural) \cong ((N_\bullet/A_\bullet)_p^\natural)^{(1)}.$$

Therefore  $I(b^{\otimes p}) \circ I(a^{\otimes p})$  is also a quasiisomorphism, and by Lemma 5.18 (ii), it suffices to prove the claim for the bimodule  $M'_\bullet$ . In other words, we may assume right away that the bimodule  $M_\bullet$  is finite semifree.

Let  $F^\bullet$  be the filtration on  $M_\bullet$  such that  $\mathrm{gr}_F^\bullet M_\bullet$  is finite free. Then  $F^\bullet$  induces a  $\mathbb{Z}/p\mathbb{Z}$ -invariant filtration  $F^\bullet$  on the tensor power  $M^{\otimes_{RP}}$ , hence on  $(M_\bullet/A_\bullet)_p^\natural$ , and by Lemma 5.18 (i), it suffices to prove that  $\mathrm{gr}_F^i(M_\bullet/A_\bullet)_p^\natural$  is convergent for any integer  $i$ . Since a direct summand of a convergent complex is obviously convergent, it in fact suffices to prove that

$$\mathrm{gr}_F^\bullet(M_\bullet/A_\bullet)_p^\natural \cong (\mathrm{gr}_F^\bullet M_\bullet/A_\bullet)_p^\natural$$

is convergent — that is, we have to prove the claim for  $\mathrm{gr}_F^\bullet M_\bullet$  instead of  $M_\bullet$ . In other words, we may assume right away that  $M_\bullet$  is finite free.

In this case, the bimodule  $CH_\bullet^p(A_\bullet, M_\bullet)$  considered as a complex in  $C_\bullet(R[\mathbb{Z}/p\mathbb{Z}])$  is chain-homotopy equivalent to its degree-0 homology placed in degree 0, so that  $(M_\bullet/A_\bullet)_p^\natural$  is convergent by Lemma 5.16.  $\square$

*Proof of Theorem 6.7.* (iii) is obvious: under the assumptions, the complex  $A_\bullet^\natural$  in  $\mathrm{Fun}(\Lambda, R)$  is trivial in negative homological degrees, so that the map  $L$  is actually an isomorphism. As in the proof of Theorem 6.5, when proving the other claims, we may assume right away that  $R$  is a field, and if  $R$  contains  $\mathbb{Q}$ , everything immediately follows from Lemma 6.3. Thus we may assume that  $R$  is annihilated by a prime  $p$ . Then (i) immediately follows from Corollary 5.12, and (ii) immediately follows from Corollary 5.12 and Corollary 6.13. For (iv), apply Lemma 6.15, (6.18), and Lemma 5.15.  $\square$

**6.5 Comparison with de Rham cohomology.** Now assume given a unital associative flat algebra  $A$  over a commutative ring  $k$ , and assume further that  $A$  is commutative. In this setting, the classic theorem of Hochschild, Kostant and Rosenberg asserts the following.

**Theorem 6.16 (Hochschild-Kostant-Rosenberg).** *In the assumptions above, assume further that  $A$  is finitely generated, and  $X = \operatorname{Spec} A$  is smooth over  $k$ . Then for any integer  $i$ , we have a natural isomorphism*

$$(6.19) \quad HH_i(A) \cong \Omega^i(A),$$

where  $\Omega^i(A) = H^0(X, \Omega_X^i)$  is the space of global differential  $i$ -forms on  $X$ .  $\square$

This results requires no assumptions whatsoever on  $k$  — in particular, it can be a field of positive characteristic, or a ring such as  $\mathbb{Z}$ . Moreover, the Hochschild-to-cyclic spectral sequence induces a differential  $B : HH_\bullet(A) \rightarrow HH_{\bullet+1}(A)$ , and in the Hochschild-Kostant-Rosenberg situation, it coincides with de Rham differential (see e.g. [Ka3, Theorem 2.2] for a short proof).

If  $k$  contains  $\mathbb{Q}$ , then one can do much more: the isomorphisms (6.19) on individual Hochschild homology groups can be lifted to the whole Hochschild homology complex  $CH_\bullet(A)$ , and they are compatible with the Connes-Tsygan differential  $B$ . As the result, one obtains a natural quasiisomorphism

$$(6.20) \quad HP_\bullet(A) \cong H_{DR}^\bullet(A)((u)),$$

where as always,  $u$  is a formal generator of cohomological degree 2.

In positive characteristic, the situation is not as good. The standard isomorphisms (6.19) typically do not lift to  $CH_\bullet(A)$ . Somewhat surprisingly, if the algebra  $A$  is a polynomial algebra in several generators, then there is a different system of isomorphisms that does lift to the level of complexes in a way compatible with  $B$  (see [L, Remark 3.2.3] and [L, Theorem 3.2.5] — in fact, on the level of homology groups, these isomorphisms are inverse to those of (6.19)). Therefore if  $A = k[x_1, \dots, x_n]$ , we do have the identification (6.20). However, the isomorphism depends not only on the algebra  $A$  but on the actual choice of the generators  $x_1, \dots, x_n \in A$ , and it is not known whether it holds if we only assume that  $\operatorname{Spec} A$  is smooth.

If the algebra  $A$  is still commutative but  $\operatorname{Spec} A$  is no longer smooth, then one can always find a simplicial resolution of  $A$  — that is, a functor  $A_\bullet$  from  $\Delta^o$  to smooth commutative  $k$ -algebras such that the  $H_i(\Delta^o, A_\bullet) = 0$  for  $i \geq 1$ , and  $H_0(\Delta^o, A_\bullet)$  is identified with  $A$  by means of an algebra map



$A_0 \rightarrow A$ . In this case, one can take the de Rham complex termwise, and obtain a complex  $\Omega^\bullet(A_\bullet)$  in the category  $\text{Fun}(\Delta^o, k)$ . Taking the standard complex of the simplicial complex, as in (3.3), we obtain a bicomplex that we also denote  $\Omega^\bullet(A_\bullet)$  by abuse of notation.

The *derived de Rham cohomology groups*  $H_{DR}^\bullet(A)$  of Illusie are then the cohomology groups of the product-total complex  $\text{Tot}(\Omega^\bullet(A_\bullet))$ . They do not depend on the choice of a resolution  $A_\bullet$ , there is a comparison theorem relating them to cristalline cohomology of  $X = \text{Spec } A$ , and they are related to the derived exterior powers of the cotangent complex  $\Omega_\bullet(A)$  by a spectral sequence of a Hodge-to-de Rham type.

One can also consider the cohomology groups  $\overline{H}_{DR}^\bullet(A)$  of the sum-total complex  $\text{tot}(\Omega^\bullet(A_\bullet))$  and call them, for example, *restricted derived de Rham cohomology groups*. Then it has been shown by Bhatt [Bh], following Beilinson, that if the base ring  $k$  is an algebra over  $\mathbb{Z}/p^n$  for some  $n$ , then  $\overline{H}_{DR}^\bullet(A)$  also do not depend on the resolution  $A_\bullet$  and satisfy even better compatibility results with respect to cristalline cohomology. Moreover, if  $k$  is an algebra over the prime field  $\mathbb{F}_p$ , then  $\overline{H}_{DR}^\bullet(A)$  are related to the derived exterior powers of  $\Omega_\bullet(A_\bullet)$  by a spectral sequence of the conjugate type.

It would be natural to expect that our theory is parallel to that of Beilinson and Bhatt, in that we have a version of the isomorphism (6.20) relating co-periodic cyclic homology and restricted de Rham cohomology, and our conjugate spectral sequence (6.6) recovers Bhatt's conjugate spectral sequence.

Unfortunately, at present, such a picture seems beyond our reach. Indeed, one can always choose a resolution  $A_\bullet$  so that its terms  $A_i$  are polynomial algebras over  $k$ , and if  $A$  is finitely generated, then one can further arrange so that all the  $A_i$  are finitely generated. Therefore for each individual  $A_i$ , we do have the isomorphism (6.20). However, these isomorphisms depend on the choice of generators, and therefore do not necessarily patch together with respect to the face and degeneracy maps in the simplicial algebra  $A_\bullet$ .

One example where Beilinson and Bhatt's approach is particularly successful — in fact, the motivating example for their whole theory — is the ring of integers  $\overline{\mathbb{Z}}_p \subset \overline{\mathbb{Q}}_p$  in the algebraic closure  $\overline{\mathbb{Q}}_p$ . This is not annihilated by a power of  $p$ , so one has to modify the definition slightly — one considers the quotient algebras  $A_n = \overline{\mathbb{Z}}_p/p^n$ ,  $n \geq 1$ , and one defines the  $p$ -adically completed cohomology by the (derived) inverse limit

$$\hat{H}_{DR}^\bullet(\mathbb{Z}_p) = \varprojlim_n \overline{H}_{DR}^\bullet(A_n).$$

In this case, Bhatt proves [Bh, Proposition 9.9] that  $\widehat{H}_{DR}^i(\overline{\mathbb{Z}}_p)$  vanishes for  $i \neq 0$ , and we have

$$\widehat{H}_{DR}^0(\overline{\mathbb{Z}}_p) = A_{cris},$$

the Fontaine’s cristalline period ring. The usual derived de Rham cohomology  $H_{DR}^\bullet(\overline{\mathbb{Z}}_p)$ , similarly  $p$ -adically completed, recovers the bigger de Rham period ring  $A_{dR}$ .

We would venture to suggest that for dimension reasons, in this particular case  $A = \overline{\mathbb{Z}}_p/p^n$ , an isomorphism (6.20) ought to exist, and we have a natural identification

$$\overline{HP}_\bullet(A) \cong \overline{H}_{DR}^\bullet(A)((u^{-1})).$$

Then by Bhatt’s result, the derived inverse limit of the groups  $\overline{HP}_\bullet(A_n)$  should be computable in terms of Fontaine’s ring. However, given our existing technology, we do not see any way to prove it.

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